Lecture 9

3. Infinite-volume measures.

Fix $q \geq 1$. $G = (V, E)$

We have monotonicity in p:

if $p \leq p'$, then

$$\varphi^{\xi}_{G,p,q} \leq_{st} \varphi^{\xi}_{G,p',q}$$

Recall:

$p \in [0,1]$ – density of edges

q – weight for every cluster

$$\varphi^{\xi}_{G,p,q}(\omega) = \frac{1}{Z^{\xi}_{G,p,q}} \cdot p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)}$$

open | closed | clusters

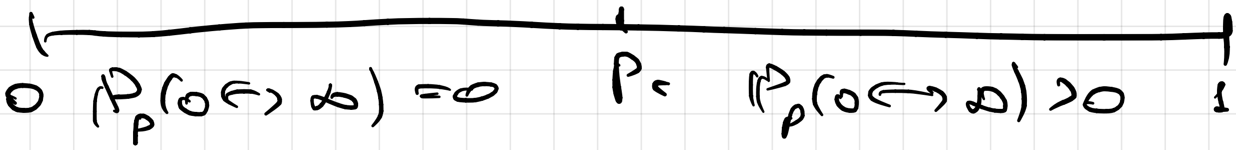
Stochastic domination:

for every $A \in \{0,1\}^E$ increasing

$$\varphi^{\xi}_{G,p,q}(A) \leq \varphi^{\xi}_{G,p',q}(A).$$

Is there a phase transition?

In percolation?



To talk about infinite clusters, we need to extend the measure to perc. conf. on $\mathbb{Z}^d$!

Proposition:
Let $q \geq 1$, $p \in [0,1]$, $G_n \nearrow \mathbb{Z}^d$ - finite domains.
Then, the following weak limits exist:

$$\varphi^0_{p,q} := \lim_{n\to\infty} \varphi^0_{G_n,p,q}$$

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Moreover, $\varphi^0_{p,q}$ and $\varphi^1_{p,q}$ are translation invariant and ergodic.

Proof:
Existence and translational invariance – in exercises.

Sketch:

1) increasing events:

(FKG) $\quad \varphi^0_{G_n,p,q}(A) \nearrow \varphi^0_{p,q}(A)$

2) events in $\{0,1\}^E$ E-finite: linear combination of increasing events

3) Carathéodory's extension thm: we can extend $\varphi^0_{p,q}$ to the σ-algebra on $\{0,1\}^{\mathbb{Z}^d}$.

4) Transl. inv.:

$\varphi^0_{p,q}$ – limit of $\varphi_{G_n,p,q}$

$\varphi^0_{p,q}$ – limit of $\varphi^0_{\tau(G_n),p,q}$

$\tau_x \circ \varphi^0_{p,q}$

Hence $\varphi^0_{p,q} = \tau_x \circ \varphi^0_{p,q}$.

Recall:

τ_x – shift by $x \in \mathbb{Z}^d$.

Remains to show ergodicity.
That is, for any $A \in \mathcal{F}$
that is translation invariant
we need to show:

$$\varphi^0_{p,q}(A) \in \{0,1\}$$

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We consider only $\varphi^1_{p,q} = \varphi$

$\boxed{\text{Exercise}}$: do the same for $\varphi^0_{p,q}$.

Step 1: mixing property:

$A, B \in \{0,1\}^E$, E - finite.

Show $$\varphi^1(A \cap T_x B) \xrightarrow[\|x\| \to \infty]{?} \varphi^1(A) \cdot \varphi^1(B).$$

Claim: Enough to prove only for increasing events.

$\boxed{\text{Exercise}}$

Recall in percolation:

If $|x| > \mathrm{diam}(E)$, then A and $T_x B$ are independent.

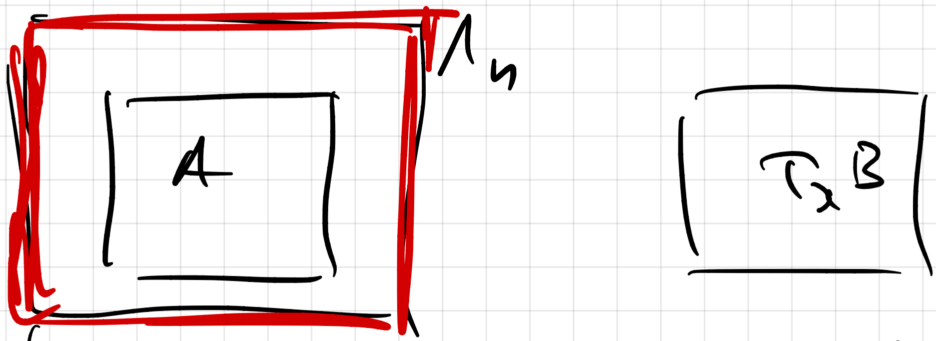
Now, assume that $A, B \uparrow$.

By the FKG:

$$\varphi^1(A \cap T_x B) \geq \varphi^1(A) \cdot \underbrace{\varphi^1(T_x B)}_{\text{transl. inv.} \; = \; \varphi^1(B)}$$

Take any n, s.t. $\underset{\text{support of } A, B}{E} \subset \Lambda_n := [-n, n]^d$

Clearly, $\Lambda_n \cap T_x \Lambda_n = \emptyset$, if $x \geq 2n$.



Use monotonicity in b.c.:

$$\varphi^1_{p,q}(A \cap T_x B) = \underbrace{\varphi^1_{p,q}(T_x B)}_{= \; \varphi^1_{p,q}(B)} \; \underbrace{\varphi^1_{p,q}(A \mid T_x B)}_{\geq \; \varphi^1_{\Lambda_n, p, q}(A)}$$

transl. inv.

$\boxed{\text{Exercise}}$

So we get

$$\varphi^{1}_{p,q}(A \cap T_x B) \le \varphi^{1}_{p,q}(B)\, \underbrace{\varphi^{1}_{\Lambda_n,p,q}(A)}_{\downarrow \ \varphi^{1}_{p,q}(A)}$$

Letting $n \to \infty$, $\|x\| \ge 2n$, we get

$$\limsup_{\|x\| \to \infty} \varphi^{1}_{p,q}(A \cap T_x B) \le \varphi^{1}_{p,q}(A) \cdot \varphi^{1}_{p,q}(B)$$

Hence

$$\varphi^{1}_{p,q}(A \cap T_x B) \to \varphi^{1}_{p,q}(A)\, \varphi^{1}_{p,q}(B)$$

Step 2: from mixing to ergodicity

Exactly as in percolation.

1) Let A be transl. inv.
2) Approximate with A' that has a finite support.
3) Look at A' and $T_x A'$.

$$\varphi^{1}_{p,q}(A' \cap T_x A') \to \varphi^{1}_{p,q}(A') \cdot \varphi^{1}_{p,q}(T_x A')$$

$$\varphi^{1}_{p,q}(A') \overset{?}{} \quad \boxed{\text{Exercise}} \quad \overset{?}{} (\varphi^{1}_{p,q}(A'))^2$$

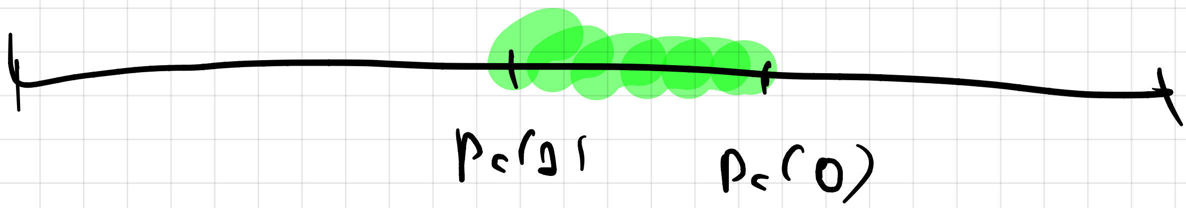
NB: we can have $\varphi^0_{p,q} \neq \varphi^1_{p,q}$!

So we can have

$$\varphi^0_{p,q}(\exists \text{ inf. cluster}) = 0$$

$$\varphi^1_{p,q}(\exists \text{ inf. cluster}) = 1$$

So a priori we could have two transitions:



When $p < p_c(0)$: $\varphi^0_{p,q}(\exists \text{ inf. clust.}) = 0$

$p > p_c(1)$: $\varphi^1_{p,q}(\exists \text{ inf. clust.}) = 1$

Next statement implies that $p_c(0) = p_c(1)$

Proposition (90s)

Let $q \geq 1$. Then the set of p, for which $\varphi^0_{p,q} \neq \varphi^1_{p,q}$ is at most countable.

Proof:

Step 1

$$\varphi^0_{p,q} = \varphi^1_{p,q} \iff \underset{\mathbb{E}^0_{p,q}(\omega_e)}{\varphi^0_{p,q}(\omega_e)} = \varphi^1_{p,q}(\omega_e)$$

for $\forall e \in E$

Proof:

"$\Rightarrow$" trivial

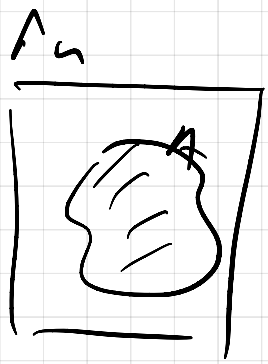
"$\Leftarrow$" Use the coupling!

Let A be increasing, with finite support $|E| < \infty$.

Take large $n > 0$.

Remember that we have coupling $\mathbb{P}_n$ of

- $\omega \sim \varphi^0_{\Lambda_n, p,q}$
- $\tilde{\omega} \sim \varphi^1_{\Lambda_n, p,q}$



s.t. $\mathbb{P}_p(\omega \leq \tilde{\omega}) = 1$.

Then,

$$0 \underset{\text{FKG}}{\leq} \varphi^1_{\Lambda_n,p,q}(A) - \varphi^0_{\Lambda_n,p,q}(A)$$

$$= \mathbb{P}_n(\tilde{\omega} \in A, \omega \notin A)$$

$$\leq \sum_{e \in E} \mathbb{P}_n(\tilde{\omega}_e = 1, \omega_e = 0)$$

Note that

$$\mathbb{P}_n(\tilde{\omega}_e = 1, \ \omega_e = 0) = \underbrace{\varphi^1_{\Lambda_n}(\omega_e = 1) - \varphi^0_{\Lambda_n}(\omega_e = 1)}_{\searrow_0 \ n \to \infty}$$

Moreover, since E is finite, also

$$\sum_{e \in E} \left(\varphi^1_{\Lambda_n}(\omega_e = 1) - \varphi^0_{\Lambda_n}(\omega_e = 1) \right) \xrightarrow[n \to \infty]{} 0.$$

Hence, $\varphi^1_{\Lambda_n, p, q}(A) - \varphi^0_{\Lambda_n, p, q}(A) \xrightarrow[n \to \infty]{} 0$

$$\varphi^1_{p,q}(A) = \varphi^0_{p,q}(A)$$

So $\varphi^1_{p,q}$ and $\varphi^0_{p,q}$ agree on increasing events with finite support.

These events generate the σ-algebra

$$\Longrightarrow \varphi^1_{p,q} = \varphi^0_{p,q}.$$

$\boxtimes$

Step 2: Enough to show that
$$\varphi^1_{p,q}(\omega_e) = \varphi^0_{p,q}(\omega_e)$$
at every continuity point of $p \mapsto \varphi^1_{p,q}(\omega_e)$.

<u>Proof:</u>

This function is increasing. Hence, it has at most countable number of discontinuity points. $\boxtimes$

Step 3:

If p is a continuity point of $\varphi^1_{p,q}(\omega_e)$, then
$$\varphi^1_{p,q}(\omega_e) = \varphi^0_{p,q}(\omega_e).$$

<u>Proof:</u>

Fix p and $\tilde{p} < p$. Take

$a := \varphi^0_{p,q}(\omega_e)$

$b := \varphi^1_{\tilde{p},q}(\omega_e) \xrightarrow[\tilde{p} \to p]{} \varphi^1_{p,q}(\omega_e)$ p - continuity point of φ

So it's enough to show:

$$a \geq b.$$

Compare p and $\tilde{p}$: (Radon-Nikodym)

Take any rand. var. X.

$$\sum_{\omega \in \{0,1\}^E} X(\omega)\, \tilde{p}^{\,o(\omega)} (1-\tilde{p})^{c(\omega)} q^{k(\omega)}$$

$$= (1-\tilde{p})^{|E|} \sum_{\omega \in \{0,1\}^E} X(\omega) \left(\frac{\tilde{p}}{1-\tilde{p}}\right)^{o(\omega)} q^{k(\omega)}$$

$$= (1-\tilde{p})^{|E|} \sum_{\omega \in \{0,1\}^E} X(\omega) \underbrace{\left(\frac{\tilde{p}}{1-\tilde{p}} \cdot \frac{1-p}{p}\right)}_{\lambda < 1}^{o(\omega)} \left(\frac{p}{1-p}\right)^{o(\omega)} \cdot q^{k(\omega)}$$

$$= \underbrace{\left(\frac{1-\tilde{p}}{1-p}\right)^{|E|}}_{c_n} \sum_{\omega \in \{0,1\}^E} X(\omega)\, \lambda^{o(\omega)} p^{o(\omega)} (1-p)^{c(\omega)} \cdot q^{k(\omega)}$$

In conclusion:

$$\varphi^{1}_{\Lambda_n, \tilde{p}, q}(X) \cdot Z^{1}_{\Lambda_n, \tilde{p}, q}$$

$$= c_n \cdot \varphi^{1}_{\Lambda_n, p, q}\left(X \cdot \lambda^{o(\omega)}\right) \cdot Z^{1}_{\Lambda_n, p, q}$$

Now take $X \equiv 1$.

$$Z^{1}_{\Lambda_n, \tilde{P}, q} = c_n \cdot Z^{1}_{\Lambda_n, P, q} \cdot \varphi_{\Lambda_n, P, q}\left(\lambda^{o(\omega)}\right)$$

Overall:

$$\boxed{\varphi^{1}_{\Lambda_n, \tilde{P}, q}(X) = \frac{\varphi^{1}_{\Lambda_n, P, q}\left(X \cdot \lambda^{o(\omega)}\right)}{\varphi^{1}_{\Lambda_n, P, q}\left(\lambda^{o(\omega)}\right)}}$$

Next time:

use this identity with

$$X = \mathbb{1}_{o(\omega) \geq (b-\varepsilon)|E_n|}$$

to prove that

$$\begin{array}{ccc} b & \leq & a \\ \shortparallel & & \shortparallel \\ \varphi^{1}_{\tilde{P}, q}(\omega_e) & & \varphi^{0}_{P, q}(\omega_e) \end{array}$$