Lecture 10

4. Phase transition

Thm

1) In $d=1$: no percolation at any $p<1$.
2) In $d \geq 2$: Let $q \geq 1$. Then the FK perc. undergoes a phase transition: there exists $0<p_c<1$

$p<p_c$	$p>p_c$
$\varphi^1_{p,q}(0 \leftrightarrow \infty) = 0$	$\varphi^1_{p,q}(0 \leftrightarrow \infty) > 0$
$\varphi^0_{p,q}$ — " —	$\varphi^0_{p,q}$ — " —

$0 \qquad\qquad p_c \qquad\qquad 1$

Proof:

1) Borel–Cantelli
2) $p_c > 0$:

$$\varphi^1_{p,q}(0 \leftrightarrow \infty) \leq \underbrace{\varphi^1_{p,1}}_{\text{Bernoulli perc.}}(0 \leftrightarrow \infty) = 0 \quad \text{when } p<\varepsilon.$$

$p_c < 1$: Enough to consider $d=2$. Duality. Counting arg.

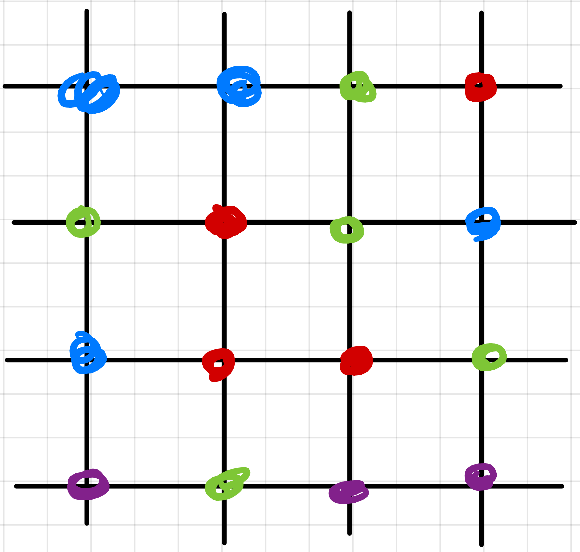
In percolation we had sharpness of the phase trans.:

$p < p_c$: $\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_k) \le e^{-ck}$, where $c > 0$ is a const.

The same is true for the FK percolation!

Before proving this, we discuss relation to the Ising and Potts models — historically, this was the original motivation.

5. Ising and Potts models



$G = (V, E) \subseteq \mathbb{Z}^d$ - finite

$q \in \mathbb{N}$, $T > 0$ temperature

Config.: $\sigma \in \{1, \dots, q\}^V$

spins / colors

q-state Potts model:

$$\mu^f_{G,T,q}(\sigma) = \frac{1}{Z_{G,T,q}} \exp\left[-\frac{1}{T} \cdot \frac{q}{q-1} \cdot \#\{u \sim v : \sigma_u \ne \sigma_v\}\right]$$

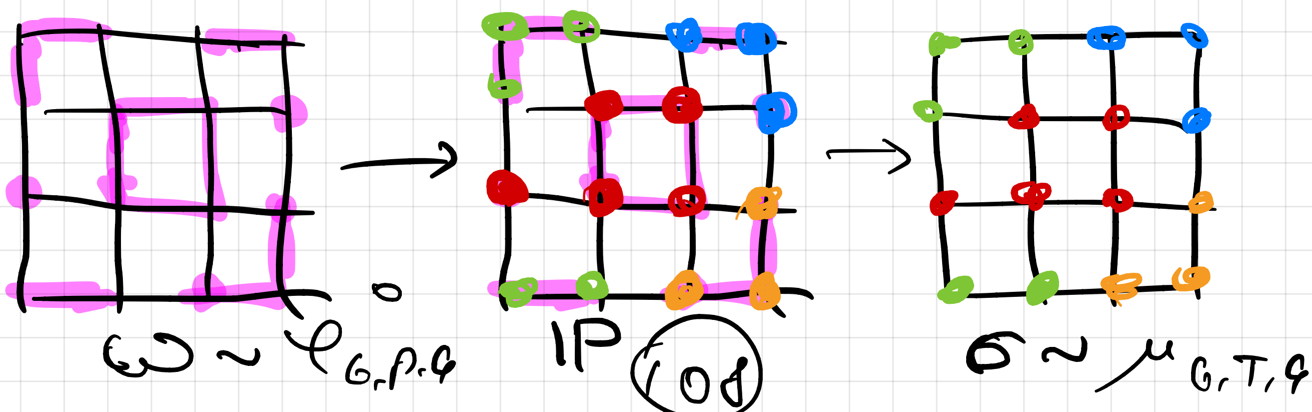
free b.c.

$q=2$: Ising model, $\sigma = \pm 1$.
Ferromagnet to paramagnet phase transition at Curie temperature, 1898

Lenz: particles are vertices crystallographic lattice
magnetic moment ± 1,
only nearest neighbors interact.

Proposition (Edwards - Sokal coupling)

Let $q \geq 2$ integer, $T > 0$.
Take
$$p = p(T) := 1 - \exp\left[-\frac{1}{T} \cdot \frac{q}{q-1}\right]$$
and $\omega \sim \varphi^0_{G,p,q}$ - random config.
Assign to every cluster of ω one of q spins uniformly indep. from other clusters.
Then, $\sigma \sim \mu^f_{G,T,q}$.



Proof:

This procedure defines a measure $\mathbb{P}$ on pairs $(\omega, \sigma) \in \{0,1\}^E \times \{1, \dots, q\}^V$.

Moreover, ω and σ must be compatible:

if $u \sim v$, then $\sigma_u = \sigma_v$.

For compatible (ω, σ) we get

$$\mathbb{P}(\omega, \sigma) = \frac{1}{Z^0_{G,p,q}} \cdot p^{o(\omega)} (1-p)^{c(\omega)} \cdot \underbrace{q^{k(\omega)} \cdot \left(\frac{1}{q}\right)^{k(\omega)}}_{=1}$$

$$= \frac{1}{Z^0_{G,p,q}} \cdot p^{o(\omega)} (1-p)^{c(\omega)}.$$

All edges uv that have $\sigma_u \neq \sigma_v$ must be closed; denote them by E_σ.

Let $\omega' := \omega|_{E \setminus E_\sigma}$

Given σ, all config. $\omega' \in \{0,1\}^{E \setminus E_\sigma}$ are possible!

give a compatible (ω, σ).

Sum over all compatible ω:

$$\sum_\omega \mathbb{P}(\omega, \sigma) = \frac{1}{Z^0_{G,p,q}} \cdot (1-p)^{|E_\sigma|} \cdot \underbrace{\sum_{\omega' \in \{0,1\}^{E \setminus E_\sigma}} p^{o(\omega')} (1-p)^{c(\omega')}}_{=}$$

$$" \prod_{e \in E \setminus E_\sigma} \underbrace{(p + (1-p))}_{1} = 1$$

In conclusion,

$$\mathbb{P}(\sigma) = \sum_{\omega} \mathbb{P}(\omega, \sigma) = \frac{1}{Z^{0}_{G,p,q}} \cdot (1-p)^{|E_\sigma|}$$

E_σ – edges of disagreement in σ!

$$1 - p = \exp\left[-\frac{1}{T} \cdot \frac{q}{q-1}\right]$$

Hence

$$\mathbb{P}(\sigma) = \frac{1}{Z^{0}_{G,p,q}} \exp\left[-\frac{1}{T} \cdot \frac{q}{q-1} \cdot \underbrace{|E_\sigma|}_{=\,|\{u \sim v : \sigma_u \neq \sigma_v\}|}\right]$$

$$= \mu^{f}_{G,T,q}(\sigma).$$

$\boxed{\checkmark}$

Corollary

This proves also that

$$Z^{@}_{G,p,q} = Z^{f}_{G,T,q}.$$

Exercises:

- sample FK from Potts?
- what do we get in Potts from (110) $\varphi^{1}_{G,p,q}$?

Corollary

$$\mu^{f}_{G,p,q}(\sigma_u = \sigma_v) = \frac{1}{q} + \frac{q-1}{q} \cdot \varphi^{0}_{G,p,q}(u \leftrightarrow v)$$

Proof:

Let $\mathbb{P}$ be the coupling. Then,

$$\mu(\sigma_u = \sigma_v) = \underbrace{\mathbb{P}(\sigma_u = \sigma_v,\ u \leftrightarrow v)}_{\mathbb{P}(u \leftrightarrow v) \,=\, \varphi^{0}_{G,p,q}(u \leftrightarrow v)} + \underbrace{\mathbb{P}(\sigma_u = \sigma_v,\ u \not\leftrightarrow v)}_{\frac{1}{q} \cdot \mathbb{P}(u \not\leftrightarrow v) \,=\, \frac{1}{q} \cdot \varphi^{0}_{G,p,q}(u \not\leftrightarrow v)}$$

$$= \frac{1}{q} + \frac{q-1}{q} \cdot \varphi^{0}_{G,p,q}(u \leftrightarrow v) \qquad \blacksquare$$

6. Sharpness of the phase transition in the FK perc.

Thm (Duminil-Copin, Raoufi, Tassion '17)

Let $q \geq 1$, $d \geq 2$. Then,

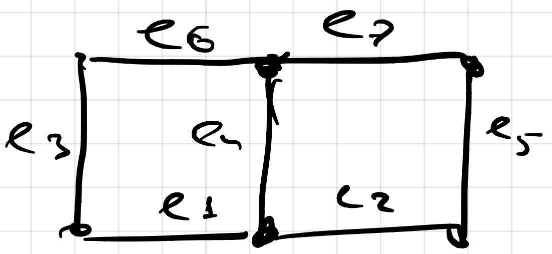
1) There exists $c > 0$, s.t. for $p > p_c$: $\varphi^1_{p,q}(0 \leftrightarrow \infty) \geq c(p - p_c)$

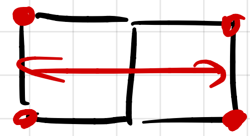
2) For $p < p_c$, there exists $c_p > 0$ s.t. $\varphi^1_{\Lambda_n, p, q}(0 \leftrightarrow \partial \Lambda_n) \leq e^{-c_p \cdot n}$.

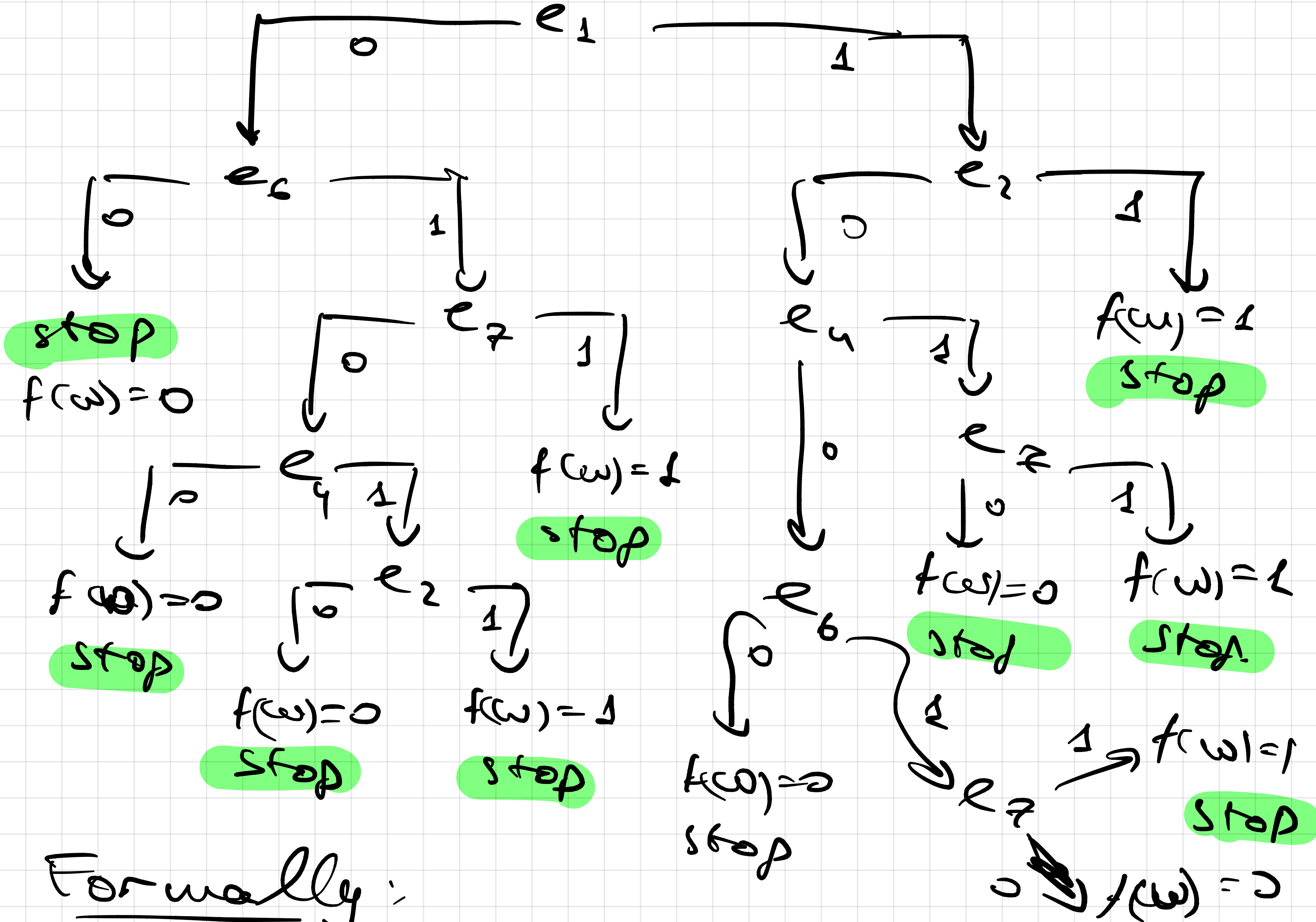
The proof is very general and goes through decision trees for a boolean function f
$(= 1_A,\ A = \{\text{crossing}\})$:

- we will be exploring the config.: reveal it edge by edge
- the next step depends on the config. revealed so far
- we stop when we know the value of f.

Q: how many edges do we have to reveal?



$f(\omega) = \mathbb{1}\{$  $\}$



Formally:

- $|E| = n$, finite set of edges.
- $e_1 \in E$ – starting edge
- for $t \le n$ – time, take

$e_{[t]} := (e_1, \ldots, e_t)$: revealed edges

$\omega_{e_{[t]}} := (\omega_{e_1}, \ldots, \omega_{e_t})$: revealed config

- ψ_t – decision rule at time t:

$\psi_t(e_{[t-1]}, \omega_{e_{[t-1]}}) = e_t \in E \setminus \{e_1, \ldots, e_{t-1}\}$

Decision tree:

$$T = (e_1, \{\varphi_t\}_{t \le n})$$

For $f : \{0,1\}^E \to \mathbb{R}$, define

$$\tau(\omega) = \tau_{f,T}(\omega)$$

$$:= \min\{t \ge 1 : \forall \omega' \in \{0,1\}^E$$

$$\underbrace{\text{if } \omega'_{[e_t]} = \omega_{[e_t]} \text{ then } f(\omega') = f(\omega)\}}$$

this means that by time t the value of f is determined.

For convenience, we will be defining T also beyond $T_T(\omega)$