Decision tree:

$$T = (e_1, \{\varphi_t\}_{t \le n})$$

For $f : \{0,1\}^E \to \mathbb{R}$, define

$$T(\omega) = T_{f,T}(\omega) := \min\{t \ge 1 : \underbrace{\forall \omega' \in \{0,1\}^E \text{ if } \omega'_{[e_t]} = \omega_{[e_t]} \text{ then } f(\omega') = f(\omega)}_{\text{this means that by time } t \text{ the value of } f \text{ is determined.}}\}$$

For convenience, we will be defining T also beyond $T_T(\omega)$

Lecture 11

Lemma 1

$$\frac{d}{dp}\varphi^1_{\Lambda_{2n},p,q}(0 \leftrightarrow \partial\Lambda_n) = \frac{1}{p(1-p)} \sum_{e \in E} \mathrm{Cov}_{\Lambda_{2n},p,q}(\mathbb{1}_{0 \leftrightarrow \partial\Lambda_n}, \omega_e)$$

Proof:

Below we fix Λ_{2n}, p, q and omit them in the writing.

For a rand. var. X, we define:

$$\mathcal{Z}(X) := \sum_{\omega \in \{0,1\}^E} X(\omega) \cdot p^{\overbrace{o(\omega)}^{\sum_e \omega_e}} (1-p)^{\overbrace{c(\omega)}^{\sum_e (1-\omega_e)}} q^{k(\omega)}$$

In particular: $\mathcal{Z}(1) = \underbrace{\mathcal{Z}}_{\text{partition func.}}$

Then,

$$\frac{d}{dp}\mathcal{Z}(X) = \sum_{\omega \in \{0,1\}^E} X(\omega)$$

$$\cdot \Big[\sum_e \omega_e \cdot \frac{1}{p} \cdot p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)}$$

$$- \sum_e (1-\omega_e) \cdot \frac{1}{1-p} \cdot p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)} \Big]$$

$$= \mathcal{Z} \cdot \sum_e \mathbb{E}\left(X \cdot \left[\frac{\omega_e}{p} - \frac{1-\omega_e}{1-p} \right] \right)$$

$$= \mathcal{Z} \cdot \frac{1}{p(1-p)} \cdot \sum_e \mathbb{E}\Big(X \cdot \big[\underbrace{\omega_e - \cancel{\omega_e p} - p + \cancel{p\,\omega_e}}_{\omega_e - p} \big] \Big)$$

Note that $\mathbb{E}(X) = \dfrac{\mathcal{Z}(X)}{\mathcal{Z}}$

Then,

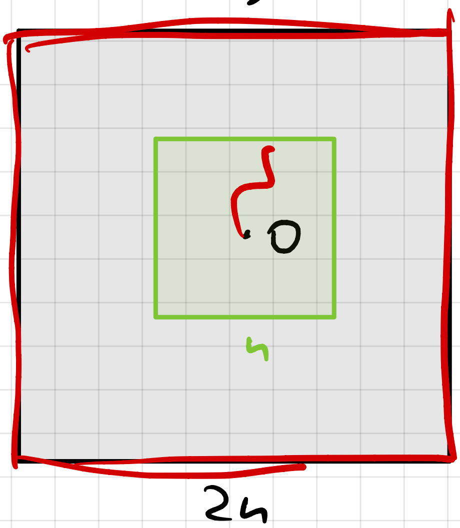
$$\frac{d}{dp}\mathbb{E}(X) = \frac{d}{dp}\left(\frac{Z(X)}{Z(1)}\right) = \frac{Z(X)'}{Z} - \frac{Z(1)'}{Z}\cdot\mathbb{E}(X)$$

$$= \frac{1}{p(1-p)}\cdot\sum_e \Big[\underbrace{\mathbb{E}(X\cdot(\omega_e-p)) - \mathbb{E}(\omega_e-p)\,\mathbb{E}(X)}_{\mathrm{Cov}(X,\,\omega_e)}\Big]$$

Take $X = \mathbb{1}_{0\leftrightarrow\partial\Lambda_n}$.

$$\theta_n(p) = \varphi^{1}_{\Lambda_{2n},p,q}(0\leftrightarrow\partial\Lambda_n)$$

$$S_n(p) = \sum_{k=0}^{n-1}\theta_k(p)$$



Lemma 2

$$\underbrace{\theta_n(1-\theta_n)}_{\mathrm{Var}(\mathbb{1}_{0\leftrightarrow\partial\Lambda_n})} \le C_{\Lambda_{2n},p,q}\cdot\frac{8S_n}{n}\cdot\sum_e \mathrm{Cov}(\mathbb{1}_{0\leftrightarrow\partial\Lambda_n},\omega_e),$$

$$C_{G,p,q} \overset{!}{=} \left[\min_{e\in E}\left(\mathrm{Var}_{G,p,q}(\omega_e)\right)\right]^{-1}$$

Thm (OSSS inequality)

Let $q \geq 1$, $p \in [0,1]$, G - finite graph.
Let $f : \{0,1\}^E \to [0,1]$ be increasing.
Fix a decision tree T for f.
Then,

$$\mathrm{Var}(f) \leq C_{G,p,q} \sum_{e \in E} \mathrm{Cov}(f, \omega_e) \cdot \delta_e(f,T),$$

where

$$\delta_e(f,T) := \varphi^1_{G,p,q}(\underbrace{\exists t \leq \tau(\omega) : e_t = e}_{\text{we reveal } e \text{ to know } f(\omega)})$$

revealment of e for T.

$C_{G,p,q}$ – as in Lemma 2.

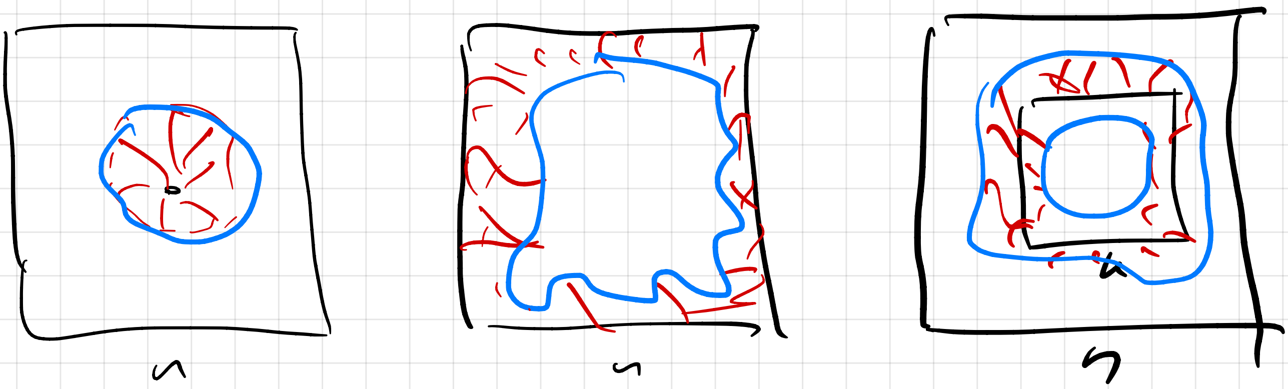
Originally found by O'Donnell, Saks, Schramm, Servedio '05 for Bernoulli percolation.
In 2017, it was extended to the FK percolation:
Duminil-Copin, Raoufi, Tassion

We'll prove it next time.

To prove Lemma 2, we need to find a tree with a small revealment — on every edge!

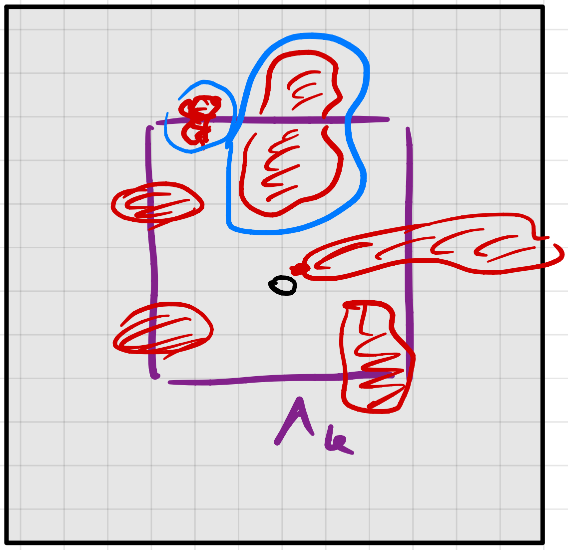
Wrong idea 1:
Explore all edges — revealment for every edg will be high.

Wrong idea 2:
Explore edges from the origin, more efficient — we explore less edges.
Problem — next to the origin we'll explore everything.
reveal
In the rest — very good.



Idea: average over all scales!

Proof:



Λ_n

Decision tree T_k: explore all clusters crossing $\partial\Lambda_k$. If one of the contains 0 and crosses $\partial\Lambda_n$, then $0 \leftrightarrow \partial\Lambda_n$.

Otherwise — there is a blocking circuit and $0 \not\leftrightarrow \partial\Lambda_n$.

It's enough to explore only edges $e = uv$, s.t. $u \leftrightarrow \partial\Lambda_k$ or $v \leftrightarrow \partial\Lambda_k$.

$$\delta_{uv}(\mathbb{1}_{0 \leftrightarrow \Lambda_n}, T_k) \leq \varphi^1_{\Lambda_{2n},p,q}(u \leftrightarrow \partial\Lambda_k) + \varphi^1_{\Lambda_{2n},p,q}(v \leftrightarrow \partial\Lambda_k)$$

By the OSSS ineq. applied for $f = \mathbb{1}_{0 \leftrightarrow \partial\Lambda_n}$ and T_k:

$$\theta_n(1-\theta_n) \leq C \cdot \sum_{e\in E} \mathrm{Cov}(\mathbb{1}_{0 \leftrightarrow \partial\Lambda_n}, \omega_e) \cdot \left(\varphi^1_{\Lambda_{2n},p,q}(u \leftrightarrow \partial\Lambda_k) + \varphi^1_{\Lambda_{2n},p,q}(v \leftrightarrow \partial\Lambda_k)\right)$$

Average over all $k = 1, 2, \ldots, n$.
On the RHS in the revealment we'll get:

$$\frac{1}{n}\sum_{k=1}^{n} \underbrace{\varphi^1_{2n}(u \leftrightarrow \partial\Lambda_k)}$$

$$\leq \varphi^1_{2n}(u \leftrightarrow \partial\Lambda_{\underbrace{|k-\|u\||}_{m}}(u))$$

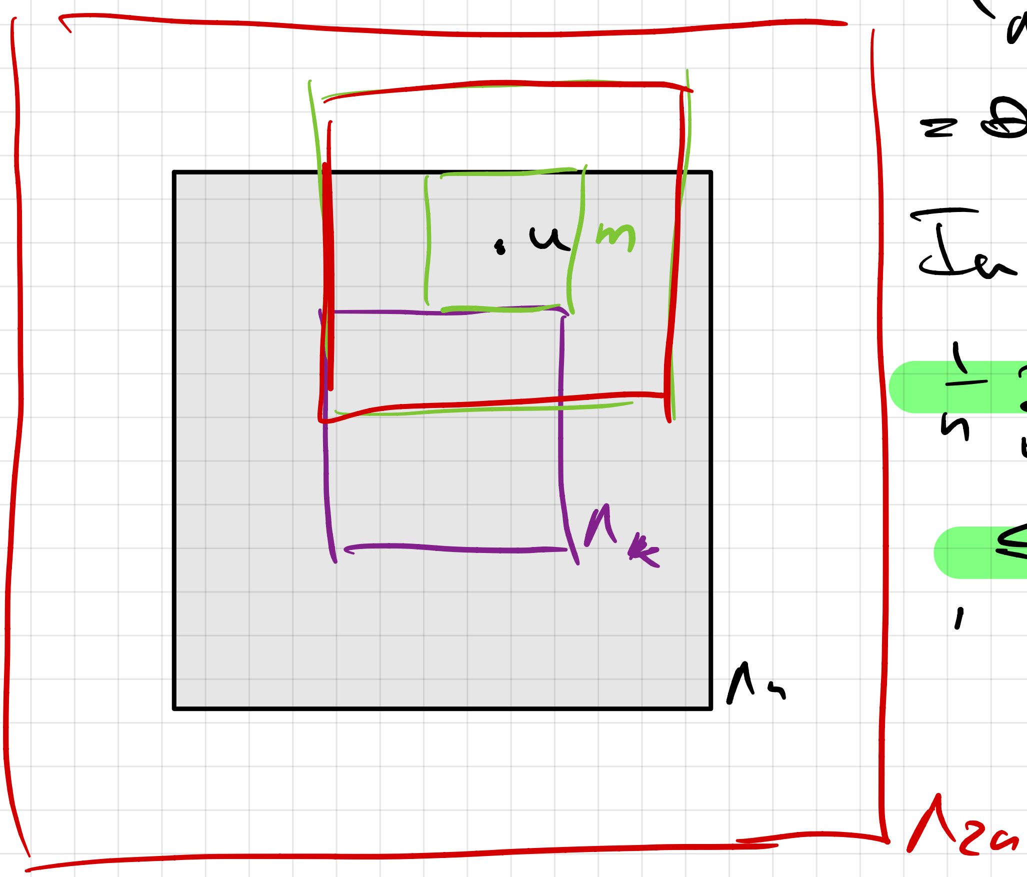
$$\leq \frac{2}{n}\sum_{m=0}^{n-1} \varphi^1_{2n}(u \leftrightarrow \partial\Lambda_m^{(u)})$$

$$\leq \frac{4}{n}\sum_{m=0}^{\frac{n-1}{2}} \cdot \underbrace{\varphi^1_{2n}(u \leftrightarrow \partial\Lambda_m(u))}$$

$$\leq \varphi^1_{\Lambda_{2m}(u)}(u \leftrightarrow \partial\Lambda_m(u))$$

$$= \varphi^1_{\Lambda_{2m}}(0 \leftrightarrow \partial\Lambda_m)$$

$$= \theta_m$$



In conclusion:

$$\frac{1}{n}\sum_{k=1}^{n} \varphi^1_{2n}(u \leftrightarrow \partial\Lambda_k)$$

$$\leq \frac{4}{n} \cdot S_n$$

(120)

Average revealment:

$$\frac{1}{n}\sum_{k=1}^{n}\left(\varphi^{1}_{2n}(u \Leftrightarrow \partial A_k) + \varphi^{1}_{2n}(v \Leftrightarrow \partial A_k)\right)$$

$\leq \frac{8}{n} \cdot S_n$ and the inequality follows

$\boxed{XC}$

Proof (Sharpness given OSSS):

Insert the equality of Lem 1 into Lem 2:

$$\Theta_n'(p) \geq \Theta_n \cdot \frac{n}{S_n} \cdot$$

$$\underbrace{\cdot (1-\Theta_n)\, \frac{1}{p(1-p)} \cdot \frac{1}{8} \cdot \min_{e \in E}(\mathrm{Var}(\omega_e))}_{\geq c(p_0, p_1) > 0}$$

whenever $p \in (p_0, p_1)$.

$0 < p_0 < p_1 < 1$

Thus, we get

$$\Theta_n' \geq c \cdot \frac{n}{S_n} \cdot \Theta_n$$

Note first that

$$\lim_{n\to\infty} \Theta_n \geq \lim_{n\to\infty} \varphi^{1}_{p,q}(0 \leftrightarrow \partial \Lambda_n)$$

$$= \varphi^{1}_{p,q}(0 \leftrightarrow \infty).$$

Also, for every $k \geq 1$,

$$\lim_{n\to\infty} \Theta_n \leq \lim_{n\to\infty} \varphi^{1}_{\Lambda_{2n},p,q}(0 \leftrightarrow \partial \Lambda_k)$$

$$= \underbrace{\varphi^{1}_{p,q}(0 \leftrightarrow \partial \Lambda_k)}_{\downarrow k\to\infty \;\; \varphi^{1}_{p,q}(0 \leftrightarrow \infty)}$$

Hence,

$$\lim_{n\to\infty} \Theta_n = \varphi^{1}_{p,q}(0 \leftrightarrow \infty)$$

The theorem follows from:

Lemma 3

Let $f_n : [0, x_0] \to [0, M]$ be increasing, differentiable and satisfy:

- $f_n \xrightarrow[n\to\infty]{} f$
- $f_n' \geq \frac{n}{\Sigma_n} \cdot f_n$, where $\Sigma_n = \sum_{k=0}^{n-1} f_k$ for all n.

then, there exists $x_1 \in [0, x_0]$,
s.t. (i) for $x < x_1$, there exists
$c_x > 0$, s.t. for any n
large enough:
$$f_n(x) \leq \exp(-c_x n)$$
(ii) for $x > x_1$,
$$f(x) \geq x - x_1.$$

This lemma can be applied for $f_n := \theta_n$. and implies:

- $p > p_c$:
$$\varphi^1_{p,q}(0 \leftrightarrow \infty) \geq c(p - p_c)$$
- $p < p_c$:
$$\theta_n \leq \exp(-c_p n)$$

Since

$$\theta_n = \varphi^1_{\Lambda_{2n}, p, q}(0 \leftrightarrow \partial\Lambda_n)$$
$$\geq \varphi^1_{\Lambda_{2n}, p, q}(0 \leftrightarrow \partial\Lambda_{2n})$$

This proves the theorem. $\boxtimes$