Lecture 12

To prove sharpness of the phase transition in the FK percolation model, it remains to prove:

- OSSS inequality
- Lemma 3 (general lemma "good diff. ineq." $\Rightarrow$ sharpness)

We start by the OSSS
Compared to Bernoulli percolation, we have dependencies,

To get closer to that setup: show how to sample any probability measure on $\{0,1\}^E$ using iid rand. variables.

Recall in perc.:
let $U_{e_1} \dots U_{e_n} \sim U[0,1]$, iid,
take $\omega_e = \begin{cases} 1 & \text{if } U_e \geq 1-p \\ 0 & \text{otherwise} \end{cases}$

Lemma 2.14

Let $U_1, \dots, U_n \sim U[0,1]$, iid

Let $(e_1, \dots, e_n)$ be a random ordering of edges in E.

Assume that: $U_1, \dots U_{t-1}$, U_t is indep. of $(e_1, \dots, e_t)$ $\forall t$.

Let μ be a probab. meas. on $\{0,1\}^E$.

Sample X - rand. var. on $\{0,1\}^E$:

$$X_{e_t} := \begin{cases} 1 & \text{if } U_t \geq \mu(\omega_{e_t} = 0 \mid \omega = X \text{ on } e_{[t-1]}) \\ 0 & \text{otherwise} \end{cases}$$

Then, $X \sim \mu$.

Rem: Values of e and U determine X

Rem: This covers an algorithm given by a decision tree:

- start by e_1 and sample X_{e_1}
- take $e_2 = \psi_2(e_1, X_{e_1})$
- sample X_{e_2},

. . . —

Proof:

Let $x \in \{0,1\}^E$ and $\bar{e}$ be the ordering of E, s.t. $\mathbb{P}(\mathbf{X} = x, \bar{\mathbf{e}} = \bar{e}) > 0$

Then:

$$\mathbb{P}(\mathbf{X} = x, \bar{\mathbf{e}} = \bar{e})$$

$$= \mathbb{P}(\mathbf{e}_1 = e_1) \cdot \mathbb{P}(\mathbf{X}_{e_1} = x_{e_1} \mid \mathbf{e}_1 = e_1)$$

$$\cdot \mathbb{P}(\mathbf{e}_2 = e_2 \mid \mathbf{e}_1 = e_1, \mathbf{X}_{e_1} = x_1)$$

$$\cdot \mathbb{P}(\mathbf{X}_{e_2} = x_{e_2} \mid \mathbf{e}_{[2]} = e_{[2]}, \mathbf{X}_{e_1} = x_1)$$

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$$= \prod_{t=1}^{n} \mathbb{P}(\mathbf{e}_t = e_t \mid \mathbf{e}_{[t-1]} = e_{[t-1]}, \mathbf{X}_{e_{[t-1]}} = x_{e_{[t-1]}})$$

(*) $$\cdot \prod_{t=1}^{n} \mathbb{P}(\mathbf{X}_{e_t} = x_{e_t} \mid \mathbf{e}_{[t]} = e_{[t]}, \mathbf{X}_{e_{[t-1]}} = x_{e_{[t-1]}})$$

We know that $\mathbf{U}_t$ is indep. of $\underbrace{\mathbf{e}_{[t]} \text{ and } \mathbf{U}_{[t-1]}}_{\text{determine } \mathbf{X}_{e_{[t-1]}}}$

Hence U_t and $X_{e_{[t-1]}}$ are indep.

Then:

$$\mathbb{P}(X_{e_t} = 1 \mid \boldsymbol{e}_{[t]} = e_{[t]}, X_{e_{[t-1]}} = x_{e_{[t-1]}})$$

$$= \mathbb{P}\Big(\{U_t \geq \mu(\omega_{e_t} = 0 \mid \omega_{e_{[t-1]}} = X_{e_{[t-1]}})\} \;\Big|\; \boldsymbol{e}_{[t]} = e_{[t]}, X_{e_{[t-1]}} = x_{e_{[t-1]}}\Big)$$

$$= \mu(\omega_{e_t} = 1 \mid \omega_{e_{[t-1]}} = x_{e_{[t-1]}})$$

Hence:

$$\mathbb{P}(X_{e_t} = x_{e_t} \mid \dots) = \mu(\omega_{e_t} = x_{e_t} \mid \dots)$$

Taking the product over t, we get:

$$(*) = \mu(\omega = x).$$

this holds for any $\bar{e}$ s.t.

$$\mathbb{P}(X = x, \bar{\boldsymbol{e}} = \bar{e}) > 0.$$

So we get:

$$\mathbb{P}(X = x, \bar{\boldsymbol{e}} = \bar{e}) = \mu(\omega = x) \cdot \prod_{t=1} \mathbb{P}(\boldsymbol{e}_t = e_t \mid \boldsymbol{e}_{[t-1]} = e_{[t-1]}, X_{e_{[t-1]}} = x_{e_{[t-1]}})$$

Fix $x \in \{0,1\}^\omega$ and sum over all orderings $\bar{e}$, s.t. $\mathbb{P}(X = x, \bar{\mathbf{e}} = \bar{e}) > 0$.

We want to show that the product sums to 1:

- for any $e_{[n-2]}$:

$$\sum_{e_{n-1} = e, f} \mathbb{P}(\mathbf{e}_{n-1} = e_{n-1} \mid \text{—"—}) = 1,$$

since we have to choose some edge.

- continue by induction

Proof (OSSS ineq. for FK perc.).

Take $\mu := \varphi^1_{G,p,q}$

As in percolation, we will need two sequences of iid rand. var.:

$$\mathbf{U}_1, \ldots, \mathbf{U}_n \sim U[0,1] \text{ iid.}$$
$$\mathbf{V}_1, \ldots, \mathbf{V}_n$$

Write $\mathbb{P}$ their joint law.

Use the decision tree

$$T = (e_1, \{\varphi_t\}_{t=1\ldots n}).$$

to construct $(\mathbf{e}, \mathbf{X})$ as in Lem 2.14: $\mathbf{X}^{2}_{\mu}$

- $\mathbf{e}_1 := e_1$
- $\mathbf{X}_{\mathbf{e}_t} := \begin{cases} 1 & \text{if } U_t \geq \mu(\omega_{e_t} = 0 \mid \omega_{e_{[t-1]}} = \mathbf{X}_{e_{[t-1]}}) \\ 0 & \text{otherwise.} \end{cases}$
- $\mathbf{e}_{t+1} = \varphi_t(\mathbf{e}_{[t]}, \mathbf{X}_{[t]}).$

As usually, we define,

$$\tau := \min\{t \geq 1 : \forall x \in \{0,1\}^E \; x_{e_{[t]}} = \mathbf{X}_{e_{[t]}} \Rightarrow f(x) = f(\mathbf{X})\}$$

time when f is computed.

Define also $\mathbf{W}^s$ obtained by swapping some entries of $\mathbf{U}$ be $\mathbf{V}$:

$$\mathbf{W}^s := (\mathbf{V}_1, \ldots, \mathbf{V}_s, \mathbf{U}_{s+1}, \ldots, \mathbf{U}_\tau, \mathbf{V}_{\tau+1}, \ldots \mathbf{V}_n)$$

Sample $\mathbf{Y}^s$ from $\mathbf{W}^s$ using Lem 2.14 and the same $\mathbf{e}$ per ω config.

$$Y^s_{e_t} := \begin{cases} 1 & \text{if } W^s \geq \mu(\omega_{e_t} = 0 \mid \dots) \\ 0 & \text{otherwise} \end{cases}$$

Note that:

- $X \sim \mu$ and is $\mathcal{U}$-meas.
- $f(X) = f(Y^0)$

  Indeed $\overline{\Phi}$ is the same and for all $t \in T$

  $$W^0_t = \mathcal{U}_t .$$

  Hence $X_t = Y^0_t$ for all $t \in T$

  Apply the def. of T.
- $Y^s \sim \mu$, for any s.
- Y^u is indep. of $\mathcal{U}$.

Now, as in percolation:

$$\mathrm{Var}(f) \leq \mu(|f - \mu(f)|)$$

$$\leq \mathbb{E}\left(\left| \mathbb{E}(f(X) \mid \mathcal{U}) - \mathbb{E}(f(Y^u) \mid \mathcal{U}) \right|\right)$$

$$= \mathbb{E}\left(\left| \mathbb{E}(f(Y^0) \mid \mathcal{U}) - \mathbb{E}(f(Y^u) \mid \mathcal{U}) \right|\right)$$

$$\leq \sum_{t=1}^{n} \mathbb{E}\left(\mathbb{E}\left(|f(Y^t) - f(Y^{t-1})| \,\middle|\, \mathcal{U}_{[t-1]} \right)\right)$$

$$\leq \sum_{e \in E} \sum_{t=1}^{n} \mathbb{E}\left(\mathbb{E}\left(|f(Y^t) - f(Y^{t-1})| \,\middle|\, \mathcal{U}_{[t-1]} \right) \mathbb{1}_{e_t = e,\, t \leq T} \right)$$

Hence, it remains to show:

$$\mathbb{E}\left(|f(Y^t) - f(Y^{t-1})| \,\middle|\, \mathcal{U}_{[t-1]} \right) \leq \frac{1}{\mathrm{Var}(\omega_e)} \cdot \mathrm{Cov}(f, \omega_e),$$

where $e_t = e$, $t \leq T$.

Since μ is FKG,

$$\underbrace{\mu(\cdot \,|\, \omega_e = 0)} \preceq \mu \preceq \underbrace{\mu(\cdot \,|\, \omega_e = 1)}.$$

Now we sample coupling using these measures

$$F_e^{\mu(\cdot|\omega_e=0)}(W^{t-1}) \leq Y^{t-1} \leq F_e^{\mu(\cdot|\omega_e=1)}(W^{t-1})$$

$$\underbrace{F_e^{\mu(\cdot|\omega_e=0)}(W^t)}_{Z} \leq Y^t \leq \underbrace{F_e^{\mu(\cdot|\omega_e=1)}(W^t)}_{Z'}$$

Then,

$$\mathbb{E}\left(|f(Y^t) - f(Y^{t-1})| \,\middle|\, \mathcal{U}_{[t-1]}\right)$$

Z and Z' are indep of $\mathcal{U}_{[t-1]}$

$$\leq \mathbb{E}\left(|f(Z) - f(Z')| \,\middle|\, \mathcal{U}_{[t-1]}\right)$$

$$= \underbrace{\mathbb{E}(f(Z'))}_{\mu(f(\omega) \mid \omega_e = 1)} - \underbrace{\mathbb{E}(f(Z))}_{\mu(f(\omega) \mid \omega_e = 0)}$$

$$= \frac{\mathrm{cov}(f, \omega_e)}{\underbrace{\mu(\omega_e) \cdot (1 - \mu(\omega_e))}_{\mathrm{Var}(\omega_e)}}.$$

$\blacksquare$