OPERATIONS ON HIGHER ALGEBRAIC K-THEORY

§ INTRODUCTION

Def: A λ-ring is the datum of a commutative unital ring R together with a family of sets maps $\lambda^k : R \to R$, $k \geq 0$ s.t.:

1) $\lambda^0(x) = 1$, $\lambda^1(x) = x$ $\forall x \in R$

2) $\lambda^k(x+y) = \lambda^k(x) + \lambda^k(y) + \sum_{i=1}^{k-1} \lambda^i(x)\lambda^{k-i}(y)$ $\forall x, y \in R$

3) Other axioms describing $\lambda^k(xy)$ and $\lambda^k(\lambda^l(x))$ in terms of polynomials in the variables $\lambda^1(x), \dots, \lambda^{kl}(x)$; $\lambda^1(y), \dots, \lambda^k(y)$ with coefficients in $\mathbb{Z}$.

FACT: $[V] \mapsto [\Lambda^k V]$ define a λ-ring structure on $K_0(X)$, X scheme ($KU(X)$ X top. space)

- For any λ-ring R we can define Adams operations $\psi^n : R \to R$:

$$\psi^1(x) = x \qquad \psi^2(x) = x^2 - 2\lambda^2(x)$$

$$\psi^k(x) = \lambda^1(x)\psi^{k-1}(x) + \dots + (-1)\lambda^{k-1}(x)\psi^1(x) + (-1)^{k-1} k \lambda^k(x)$$

They have other pleasant properties:

- Adams op. are ring homom. $\forall m,n \quad \psi^n(\psi^m) = \underset{\psi^m(\psi^n)}{\underset{\shortparallel}{\psi^{mn}}}$

- $\forall k \quad \psi^k(\ell) \simeq \ell^{\otimes k}$ $\quad \ell$ line bundle

They have been used to discover interesting things:

X top. space

- $$\bigoplus_{i\geq 0} KU(X)^{(i)}_{\mathbb{Q}} \cong KU(X)_{\mathbb{Q}} \xrightarrow[\cong]{ch} \bigoplus_{i\geq 0} \overset{\checkmark}{H}{}^{2i}(X,\mathbb{Q})$$

X scheme

- $$\bigoplus_{i\geq 0} K_0(X)^{(i)}_{\mathbb{Q}} \cong K_0(X)_{\mathbb{Q}} \xrightarrow[\cong]{ch} \bigoplus_{i\geq 0} CH^i(X)_{\mathbb{Q}}$$

Where $K_0(X)^{(i)}_{\mathbb{Q}}$ is the eigenspace of $\psi^j_{\mathbb{Q}}$ of eigenvalue j^i

Adams-Riemann-Roch : $f: X \longrightarrow Y$ projective l.c.i. morphism between (regular) schemes. Then $\psi^{\gamma}_{\mathbb{Q}}(f_*) = f_*(\psi^{\gamma}_{\mathbb{Q}} \cdot \theta^{\gamma}(f)^{-1})$, i.e. the following commutes:

$$\begin{tikzcd}
K_0(X)_{\mathbb{Q}} \arrow[r, "\theta^{\gamma}(f)^{-1} \cdot \psi^{\gamma}_{\mathbb{Q}}"] \arrow[d, "f_*"'] & K_0(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\
K_0(Y)_{\mathbb{Q}} \arrow[r, "\psi^{\gamma}_{\mathbb{Q}}"'] & K_0(Y)_{\mathbb{Q}}
\end{tikzcd}$$

* * *

In a category $\mathcal{C}$ with finite products (hence with a terminal object $*$) one can define **algebraic structures**:

- A **group object** is the datum of an obj $K \in \mathcal{C}$ and maps $m: K \times K \longrightarrow K$, $(-)^{-1}: K \longrightarrow K$, $0: * \longrightarrow K$ representing multiplication, inv. and the neutral element $\textcircled{+}$ commutative diagrams representing the axioms.

Example:

ASSOCIATIVITY

$$\begin{tikzcd}
 & K \times K \times K \arrow[dl, "m \times id_K"'] \arrow[dr, "id_K \times m"] & \\
K \times K \arrow[dr, "m"'] & & K \times K \arrow[dl, "m"] \\
 & K &
\end{tikzcd}$$

- e.g. group objects in Sets are groups.
  " " in Ho(Top$_*$) are H-groups.
- λ-ring object is a ring object K + map $\lambda^p : K \to K$ sat. certain axioms

UNSTABLE OPERATIONS ON HIGHER K-THEORY

S regular affine base scheme.

$\mathcal{C}$: $\underset{Sm_S}{\text{smooth schemes}/S} \subseteq \underset{DSch_S}{\text{divisorial schemes}/S} \subseteq \underset{Sch_S}{\text{Noetherian schemes}/S}$

simplicial presheaves over $\mathcal{C}$ + local injective model str. w.r. to the Zariski or Nis + $\mathbb{A}^1$-localization

$\mathrm{sPre}(\mathcal{C})$ $\qquad$ $\mathrm{sPre}_{Zar(Nis)}(\mathcal{C})$ $\qquad$ $\mathrm{sPre}^{\mathbb{A}^1}(\mathcal{C})$

Homotopy cat of $\mathcal{H}_{Zar(Nis)}(\mathcal{C})$

$$\mathcal{H}(S) := \mathrm{Ho}\left(\mathrm{sPre}^{\mathbb{A}^1}_{Nis}(\mathrm{Sm}_S)\right)$$ ⟵ UNSTABLE MOTIVIC HTPY CAT.

Let K be the K-theory simplicial pres'eaf

Theorem ☆ (RIOU-CISINSKI-ZANCHETTA): For any $n, m \geq 1$ the maps in the following commutative diagram are bijections:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{H}_{Zar}(\mathrm{Sch}_S)}(K^n,K^m) \cong \mathrm{Hom}_{\mathcal{H}_{Zar}(\mathrm{DSch}_S)}(K^n,K^m) \arrow[r, "\textcircled{2}", "\cong"'] \arrow[d, "\checkmark\ \pi_0"', "\cong"] & \mathrm{Hom}_{\mathcal{H}(S)}(K^n,K^m) \arrow[d, "\pi_0", "\textcircled{1}"'] \\
\mathrm{Hom}_{\mathrm{Pre}(\mathrm{DSch}_S)}(K_0^n,K_0^m) \arrow[r, "?\ \textcircled{3}\ \cong", "res"'] & \mathrm{Hom}_{\mathrm{Pre}(\mathrm{Sm}_S)}(K_0^n,K_0^m)
\end{tikzcd}$$

The horizontal maps are induced by $\mathrm{Sm} \subseteq \mathrm{DSch}_S$ + localization!

Corollary: The algebraic structures on K in both $\mathcal{H}(S)$ and $\mathcal{H}_{Zar}(DSch_S)$ are uniquely determined by the ones you have on K_0. K has a λ-ring structure in $\mathcal{H}(S)$ and $\mathcal{H}_{Zar}(DSch_S)$.

Proof of ☆: ① is $\cong$ (Riou) + ② is $\cong$ (Zanchetta-Cisinski)
③ is 1-1 (Zanchetta)

$n = m = 1$

- ① In $\mathcal{H}(S)$ $K \cong \mathbb{Z}\times Gr \cong \operatorname{colim}_{i\in\mathbb{N}} X_i$ $\qquad X_i = \bigsqcup_i Gr_{i,i}$

$$R^1\varprojlim_{n\in\mathbb{N}} K_1(Gr_{n,n})^n \cong R^1\varprojlim_{n\in\mathbb{N}} \operatorname{Hom}_{\mathcal{H}_\bullet(S)}(S^1\wedge X_{n+1}, K)$$

$\pi_0 F : \operatorname{Hom}_{\mathcal{H}(S)}(-, F)$

$$\begin{tikzcd}
R^1\varprojlim_{n\in\mathbb{N}} K_1(Gr_{n,n})^n \arrow[d, equal] & & \\
\operatorname{Ker}\eta \arrow[r, tail] \arrow[d, phantom, "\cong 0"] & \operatorname{Hom}_{\mathcal{H}(S)}(\mathbb{Z}\times Gr, K) \arrow[r, tail, "\pi_0"] \arrow[dr, "\eta", "\cong"'] & \operatorname{Hom}_{Pre(Sm_S)}(\pi_0\mathbb{Z}\times Gr, \overset{\cong K_0}{K_0}) \arrow[d, "\tau^*_{\mathbb{Z}\times Gr}"] \\
 & & \operatorname{Hom}_{Pre(Sm_S)}(\mathbb{Z}\times Gr, K_0) \arrow[d, phantom, "\cong"] \\
 & & \varprojlim_{n\in\mathbb{N}} K_0(Gr_{n,n})^n
\end{tikzcd}$$

$\bullet \to \bullet \to \bullet$ Milnor's exact seq.

$$0 \to R^1\varprojlim_{n\in\mathbb{N}} \pi_1(X_n) \to \pi_0 X \to \varprojlim_{n\in\mathbb{N}} \pi_0(X_n) \to 0$$

$X = \varinjlim_{n\in\mathbb{N}} X_n$

- For any X in $Pre(Sm_S)$ we have a map $\tau_X : \hat{X} \to \pi_0 X$
- $\operatorname{Ker}\eta = 0$ | You can ch. show th. t $\tau_{\mathbb{Z}\times Gr}(U)$ are surjective $\forall\, U$ affine in Sm_S

$\Longrightarrow \tau^*_{\mathbb{Z}\times Gr}$ is injective

$n = m = 1$

- ② One first prove that $\mathrm{Hom}_{\mathcal{H}_{Zar}(DSch_S)}(\mathbb{Z}\times BGL, K) \longrightarrow \mathrm{Hom}_{\mathcal{H}(S)}(\mathbb{Z}\times BGL, K)$ is $\cong$. This uses the fact that $\mathbb{Z}\times BGL$ is a colimit of SMOOTH schemes (some GL_n). Then we use that $\mathbb{Z}\times BGL \to \mathbb{Z}\times BGL^{+}$ is $\cong$ in $\mathcal{H}(S)$ and it induces an isom. on $\mathrm{Hom}(-,K)$ in $\mathcal{H}_{Zar}(DSch_S)$ as it happens for the $+$ $-$ construction in Top.

- ③ We need to prove that ③ is 1-1.

Theorem (HAUSEN, BRENNER, SCHROER, ZANCHETTA): Let be $X \in DSch_S$. Then there exists $W \in Sm_S$ and a closed embedding $X \hookrightarrow W$ in $DSch_S$. ⌋

This is a fundamental ingredient for:

Theorem (ZANCHETTA): Let be $X \in DSch_S$ and $F_1, \dots, F_n \in Vect(X)$. Then there exist $Y \in Sm_S$ $\varphi : X \to Y$ and $G_1, \dots, G_n \in Vect(Y)$ s.t. $\varphi^*(G_i) \cong F_i \;\; \forall\, i = 1, \dots, n$.

$\Rightarrow$ ③ is 1-1 is an exercise now.

Rmk: The Theorem also holds for inner product spaces.

Theorem (ZANCHETTA): For any $m, n \geq 1$, the maps in the following square are bijections.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{H}_{tor}(DSch_S)}(KSp^n, KSp^m) \arrow[r, "②", "\cong"'] \arrow[d, "\pi_0"', "\cong"] & \mathrm{Hom}_{\mathcal{H}(S)}(KSp^n, KSp^m) \arrow[d, "\pi_0", "\cong ①"'] \\
\mathrm{Hom}_{Pre(DSch_S)}(KSp_0^n, KSp_0^m) \arrow[r, "③ \cong", "res"'] & \mathrm{Hom}_{Pre(Sm_S)}(KSp_0^n, KSp_0^m)
\end{tikzcd}$$

Rmk: For GW, we can show that ③ is 1-1 and that ① is $\twoheadrightarrow$.