Characterization based approach for construction of goodness-of-fit tests: randomly censored data case

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$X_1, X_2, ..., X_n$ i.i.d with d.f. F.

$$H_0(F \in \mathcal{F}_0) \text{ against } H_1(F \in \mathcal{F}_1)$$

- Glivenko-Cantelli theorem, goodness of fit test based on empirical d.f.
- Kolmogorov-Smirnov test

$$T_n = \sup_{t \in R} |F_n(t) - F_0(t)|$$

- Cramer-von Mises test

$$T_n = \int_{-\infty}^{\infty} (F_n(t) - F_0(t))^2 f_0(t) dt$$

Recently, the characterization based approach for construction of goodness-of-fit tests become very popular

- The idea originated from Yu. V. Linnik ("Linear forms and statistical tests", 1953)
- Tests based on characterizations use some intrinsic properties of distributions
- Tests are usually free of some parameter (or distribution-free)

This idea have resulted in a bunch of new goodness-of-fit tests!

- Equidistribution-type

$$\psi_1(X_1, ..., X_m) \overset{d}{=} \psi_2(X_1, ..., X_m) \text{ if and only if } X_i \sim F_0$$

- Independence-type

$$\psi_1(X_1, ..., X_m) \text{ and } \psi_2(X_1, ..., X_m) \text{ are independent}$$

$$\text{if and only if } X_i \sim F_0$$

- Functional equation-type
  - recall memoryloss property of exponential distribution
- Moment-type
- Entropy-type
- ...

$$\psi_1(X_1, ..., X_m) \stackrel{d}{=} \psi_2(X_1, ..., X_m)$$

- Desu (1971) $2\min(X, Y) \stackrel{d}{=} X$
- Puri-Rubin (1970) $|X - Y| \stackrel{d}{=} X$
- Milošević-Obradović (2016)

$$X_{(k;n)} \stackrel{d}{=} X_{(k-1;n-1)} + \frac{1}{n} X_n$$

$$X_{(k;n)} \stackrel{d}{=} X_{(k-1;n)} + \frac{1}{n-k+1} X_0$$

$$X_{(k;n)} \stackrel{d}{=} \frac{1}{n} X_1 + \frac{1}{n-1} X_2 + \cdots + \frac{1}{n-k+1} X_k$$

- Obradović et al. (2015)
  $\max\left(\frac{X}{Y},\frac{Y}{X}\right) \overset{d}{=} X$
- Volkova (2015) $\frac{X_{k,k-1}}{X_{k,k}} \overset{d}{=} X_1$

$\Bigg\}\ \mathcal{P}a(\lambda)$

- Ahsanullah and Nevzorov (2019) $\frac{aX+bY}{(\sqrt{a}+\sqrt{b})^2} \overset{d}{=} X$ $\quad\}\ \mathcal{L}(0,\sigma^2)$
- Polya (1923) $\frac{X+Y}{\sqrt{2}} \overset{d}{=} X$ $\quad\}\ \mathcal{N}(0,\sigma^2)$

- equality of distribution functions $G^{\psi_1}(x) = G^{\psi_2}(x)$
- equality of density functions $g^{\psi_1}(x) = g^{\psi_2}(x)$
- equality of different integral transforms (Laplace transforms, characteristics functions,...) $\phi^{\psi_1}(x) = \phi^{\psi_2}(x)$

Natural approach is to estimate mentioned functions and to consider their "difference" (or quasi-difference) in some space.

Many examples of such tests for the case of testing exponentiality will be presented during the next talk.
Nikitin (1996), Baringhous and Henze (2000), Henze and Meintanis (2005), Nikitin and Volkova (2010), Volkova (2010), Jovanović et al.(2015), Milošević and Obradović (2016), Milošević (2016), Cuparić et al. (2019), Jiménez-Gamero et al. (2020), Cuparić et al. (2020)

- In survival analysis, sample is often limited due to some censoring mechanism.
- The outcome variable of interest is the time until an event occurs (survival time).
- Most of the times, we have information about the survival time of the individual, but we don't know exactly the survival time.
- There is a need for studying the goodness-of-fit tests for this kind of data.

Let $X_1', ..., X_n'$ be a random variables from a non-negative continuous distribution function F.

Let $C_1, ..., C_n$ be censoring random variables from non-negative continuous distribution function G.

Sample: $(X_1, \delta_1), ..., (X_n, \delta_n)$,
where $X_i = \min\{X_i', C_i\}$ and $\delta_i = I(X_i' \leq C_i), \forall i \in \{1, ..., n\}$.

We assume the independent censoring model.

- First adaptations proposed in Koziol,Greem (1976), Barlow, Proschan (1969)
- χ^2 tests Akritas (1988)
- Although the first tests have been proposed in 20th century until now this field of research is still not well explored
- Some tests that are functional of empirical distribution function are modified with interchanging empirical distribution function with Kaplan-Meier estimator $F_n(x)$ is replaced with

$$\widehat{F}_n(x-) = 1 - \prod_{X_i < x}\left(1 - \frac{\delta_i}{\sum_{k=1}^{n} I\{X_k \geq x\}}\right)$$

  - Koziol, Green (1976)
  - Strzalkowska-Kominiak, Grané (2014)
  - ...
  - Bothma et al. (2021)
- The adaptation is done using inverse probability censoring weights (IPCW) method.
  - Kattumannil and Anisha (2019)
  - our recently proposed new exponentiality tests

- Puri-Rubin characterization $X_0 \overset{d}{=} |X_1 - X_2|$ iff $X_i \sim \mathcal{E}(\mu)$
- Desu characterization: $X_0 \overset{d}{=} m\min(X_1, ..., X_m)$ for each m iff $X_i \sim \mathcal{E}(\mu)$

Milošević and Obradović (2016), Cuparić et al. (2019a) and Cuparić et al.(2019b)

$$J_{n,a}^{\mathcal{I}} = \int_0^{\infty} \Big(\frac{1}{n}\sum_{i_1=1}^{n} e^{-tX'_{i_1}} - \frac{1}{\binom{n}{2}} \sum_{1\le i_1<i_2\le n} e^{-t\psi^{\mathcal{I}}(X'_{i_1},X'_{i_2})}\Big) e^{-at}dt,$$

$$M_{n,a}^{\mathcal{I}} = \int_0^{\infty} \Big(\frac{1}{n}\sum_{i_1=1}^{n} e^{-tX'_{i_1}} - \frac{1}{\binom{n}{2}} \sum_{1\le i_1<i_2\le n} e^{-t\psi^{\mathcal{I}}(X'_{i_1},X'_{i_2})}\Big)^2 e^{-at}dt,$$

where $\mathcal{I} \in \{\mathcal{P}, \mathcal{D}\}$ indicates the characterization the test is based on. In the expressions above $\psi^{\mathcal{P}}(x, y) = |x - y|$, its Laplace transform is $Ee^{t|X-Y|}$, and $\psi^{\mathcal{D}}(x, y) = 2\min(x, y)$ with Laplace transform $Ee^{t\cdot 2\min(X,Y)}$

IPCW U-empirical process

$$U_{n,c}(t) = \frac{1}{\binom{n}{2}} \sum_{i<j} \frac{h(X_i, X_j; t)\delta_i\delta_j}{K_c(X_i-)K_c(X_j-)} = \frac{1}{\binom{n}{2}} \sum_{i<j} H((X_i, \delta_i), (X_j, \delta_j); t)$$

where $K_c(x-) = 1 - G(x-) = P\{C \geq x\}$ is the survival function of the censoring variable C such that $K_c(x) > 0$, for each x, with probability 1.

Then, our test statistics are

$$J^{\mathcal{I}}_{c,a} = \int_0^\infty U^{\mathcal{I}}_c(t)e^{-at}dt$$

$$M^{\mathcal{I}}_{c,a} = \int_0^\infty (U^{\mathcal{I}}_c(t))^2 e^{-at}dt$$

If the survival distribution of censoring variable is unknown, K_c have to be replaced by its consistent estimator. We use the Kaplan-Meier estimator $\widehat{K}_c(x-) = \prod_{X_i<x}\left(1 - \frac{1-\delta_i}{\sum_{k=1}^{n} I\{X_k \geq X_i\}}\right)$. Now

$$\widehat{U}_{n,c}(t) = \frac{1}{\binom{n}{2}} \sum_{i<j} \frac{h(X_i, X_j; t)\delta_i\delta_j}{\widehat{K}_c(X_i-)\widehat{K}_c(X_j-)},$$

for fixed t, is not actually $U-$statistic because $\widehat{K}_c$ uses whole sample.

Then our statistics become

$$\widehat{J}^{\mathcal{I}}_{c,a} = \int_0^\infty \widehat{U}^{\mathcal{I}}_c(t) e^{-at} dt$$

$$\widehat{M}^{\mathcal{I}}_{c,a} = \int_0^\infty (\widehat{U}^{\mathcal{I}}_c(t))^2 e^{-at} dt$$

The IPCW based V-statistics are equal to Kaplan-Meier counterpart when the largest observation is not censored.

Theorem

Let $X_1', ..., X_n'$ be the sample of i.i.d. random variables with distribution function $F(x) = 1 - e^{-\frac{x}{\mu}}$, $x > 0$, and $(X_1, \delta_1), ...(X_n, \delta_n)$ the corresponding censored sample. Suppose that

$$a) \int_0^\infty \frac{1}{K_c(u-)} dF(u) < \infty, \qquad b) \int_0^\infty \frac{1-F(u)}{K_c^2(u)} dG(u) < \infty,$$

$$c) \int_0^\infty \frac{u^2}{K_c(u-)} dF(u) < \infty, \qquad d) \int_0^\infty \frac{u^2(1-F(u))}{K_c^2(u)} dG(u) < \infty.$$

Then, $\{\sqrt{n}\widehat{U}_c(t)\}$ converges in $C(\mathbb{R}^+)$ to some centered Gaussian process $\{\eta(t)\}$ with covariance equal to

$$cov(\eta(t_1), \eta(t_2)) = 4E(\zeta(t_1)\zeta(t_2)),\ t_1, t_2 \in \mathbb{R}^+,$$

where $\zeta(t_1) = \frac{h_1(X_1;t_1)\delta_1}{K_c(X_1-)} + \int_0^\infty \omega(u; t_1) dM_1^c(u)$ and $h_1(x;t) = E(h(X_1', X_2'; t) | X_1' = x)$ is the first projection of kernel h, for $u \geq 0$

$$\omega(u;t) = \frac{1}{P(X \geq u)} \int_u^\infty h_1(x;t) dF(x)$$

and $dM_1^c(t) = dN_1^c(t) - Y_1(t) d\Lambda_c(t)$, $N_1^c(t) = I\{X_1 \leq t, \delta_1 = 0\}$, and $Y_i(t) = I\{X_i \geq t\}$, and $\Lambda_c(t) = -\log K_c(t)$.

$$\widehat{J}^{\mathcal{I}}_{c,a} = \int\limits_0^\infty \widehat{U}^{\mathcal{I}}_c(t)e^{-at}dt$$
$$= \frac{1}{\binom{n}{2}}\sum_{i<j}\frac{1}{2}\left(\frac{1}{a+X_i}+\frac{1}{a+X_j}-\frac{2}{a+\psi^{\mathcal{I}}(X_i,X_j)}\right)\frac{\delta_i\delta_j}{\widehat{K}_c(X_i-)\widehat{K}_c(X_j-)}$$

Theorem

Suppose that conditions of the Theorem 1 holds. Then $\sqrt{n}\widehat{J}_{c,a} \overset{D}{\to} \mathcal{N}(0,\sigma^2)$, *where*

$$\sigma^2 = 4Var\left(\frac{\Phi_1(X_1)\delta_1}{K_c(X_1-)} + \int\limits_0^\infty \frac{1}{P(X\geq u)}\int_u^\infty \Phi_1(x)dF(x)dM_1^c(u)\right),$$

and $\Phi_1(x) = E(\frac{1}{a+x}+\frac{1}{a+X_2'}-\frac{2}{a+\psi^{\mathcal{I}}(x,X_2')})$ *is the first projection of kernel.*

$$\widehat{M}^{\mathcal{I}}_{c,a} = \int\limits_0^\infty (\widehat{U}^{\mathcal{I}}_c(t))^2 e^{-at} dt$$
$$= \int\limits_0^\infty \left(\frac{1}{\binom{n}{2}} \sum_{i<j} \frac{\left(e^{-tX_i} + e^{-tX_j} - 2e^{-t\psi^{\mathcal{I}}(X_i,X_j)}\right)\delta_i\delta_j}{2\widehat{K}_c(X_i-)\widehat{K}_c(X_j-)} \right)^2 e^{-at} dt,$$

Theorem

Under the conditions of Theorem 1, it follows that

$$n\int_0^\infty \widehat{U}_c^2(t)e^{-at}dt \xrightarrow{D} \sum_{k=1}^\infty \upsilon_k W_k^2,$$

where $\upsilon_k, k = 1, 2, ...,$ *is the sequence of eigenvalues of the integral operator* A, *defined for functions* $q \in C(\mathbb{R}^+)$ *for which* $\int_0^\infty q(t)^2 e^{-at} dt < \infty$, *by*

$$Aq(t_1) = \int_0^\infty cov(\eta(t_1), \eta(t_2))q(t_2)e^{-at_2}dt_2,$$

where $cov(\eta(t_1), \eta(t_2))$ *given in Theorem 1, and* W_k, $k = 1, 2, ...,$ *are independent standard normal variables.*

- simple and composite hypothesis
- no information about censoring distribution
- Koziol-Green model (Koziol and Green (1976)) is used for achieving certain censoring rate: failure time (random variable of interest) and censoring time intensities are proportional, i.e. $1 - G(x) = (1 - F(x))^{\beta}$, $\beta > 0$ unknown parameter
- in small and moderate sample sizes, the usage of resampling procedure is necessary (here we consider sample size $n = 50$)

- The Cramer-von Mises test (Koziol and Green (1976))

$$\omega^2 = \int_0^\infty (\widetilde{F}_n(t) - (1 - e^{-\frac{t}{\mu}}))^2 e^{-\frac{t}{\mu}} dt,$$

where $\widetilde{F}_n$ is Kaplan-Meier estimator modified with $\widetilde{F}_n(t) = 1$, for $t \geq X_{(n)}$ if the largest observation is censored.

- The χ^2 test (Akritas (1988))

simple hypothesis: $A_{nr} = \sum_{j=1}^{r} \frac{(N_{1j} - n\widehat{p}_{1j})^2}{n\widehat{p}_{1j}}$, where

$$N_{1j} = \sum_{i=1}^{n} I\{X_i \in A_j, \delta_j = 1\},$$

$$\widehat{p}_{1j} = \frac{1}{\mu} \int_{A_j} (1 - \frac{1}{n} \sum_{i=1}^{n} I\{X_i \leq x\}) dx$$

composite hypothesis: $A_{nr} = \widetilde{V}_n' A \widetilde{V}_n$, where A is generalised inverse of the matrix $\widehat{\Sigma} - \widehat{B}\widehat{I}_{\widehat{\mu}}\widehat{B}'$ with $\widehat{\Sigma} = diag(\widetilde{p}_{11}, ..., \widetilde{p}_{1r})$ and $\widehat{B}$ is vector with column elements $\widehat{b}_j = \int_{A_j} (1 - \widehat{H}(x)) dx$, and vector $\widetilde{V}_n = \frac{1}{\sqrt{n}}((N_{11} - n\widetilde{p}_{11}), ..., (N_{1r} - n\widetilde{p}_{1r}))$, $\widetilde{p}_{1j} = \frac{1}{\widehat{\mu}} \int_{A_j} (1 - \widehat{H}(x)) dx$, and $\widehat{\mu}$ is the Kaplan-Meier estimator of μ.

- The test based on maximal correlations (Strzalkowska-Kominiak and Grané (2017))

$$Q_n^S = \frac{\sqrt{n}Q_n}{\sqrt{\sigma_n^2}} = \frac{\sqrt{n}}{\sqrt{\sigma_n^2}} \sum_{i \neq j} \omega_{in}\omega_{jn}((6Y_i - 2)I\{Y_j \leq Y_i\} - 6Y_i I\{Y_j > Y_i\}),$$

  where $Y_i = 1 - e^{-\frac{X_i}{\mu}}$, and $\omega_{in} = F_n(Y_i) - F_n(Y_i -)$, and F_n is Kaplan-Meier estimator of distribution F, and σ_n^2 is consistent estimator of the variance of statistic Q_n

- The test based on properties of DMTTF class of life distributions (Kattumannil and Anisha (2019))

$$\Delta_n = \frac{1}{\binom{n}{2}} \sum_{i<j} \frac{\delta_i \delta_j}{\widehat{K}_c(X_i)\widehat{K}_c(X_j)} \Big(2\min\{X_i, X_j\} - \frac{1}{2}(X_i + X_j)\Big).$$

$H_0 : F(x) = 1 - e^{-x},\ x \geq 0$ Bootstrap algorithm

1. Based on $(X_1, \delta_1), ..., (X_n, \delta_n)$ compute the test statistic T_n ;
2. Estimate critical value $q^*_{n,1-\alpha}$:
   - a. generate sample $C^*_1, ..., C^*_n$ from Kaplan-Meier distribution function G_n;
   - b. generate new sample $X'_1, ..., X'_n$ from $\mathcal{E}(1)$ distribution;
   - c. using (a) and (b) construct bootstrap sample $(X^*_1, \delta^*_1), ..., (X^*_n, \delta^*_n)$, where $X^*_i = \min(X^{'}_i, C^*_i)$, and $\delta^*_i = I\{X^{'}_i \leq C^*_i\}$;
   - d. based on the sample from (c) determine the value of test statistic

$$T^*_n = T_n((X^*_1, \delta^*_1), ..., (X^*_n, \delta^*_n));$$

   - e. repeat steps (a)-(d) B times;
   - f. based on the obtained sequence of bootstrapped test statistics estimate critical value $q^*_{n,1-\frac{\alpha}{2}}$;
3. Reject H_0 if $T_n \geq q^*_{n,1-\alpha}$;
4. Repeat steps 1-3 N times. Estimated test power is percentage of rejected H_0.

Theorem

Assume that the conditions from previous Theorem are fulfilled. Then, conditionally on the sample $(X_1, \delta_1), ..., (X_n, \delta_n)$*, we have that the* $\{\sqrt{n}\widehat{U}_c^*(t)\}$ *converges weakly to process* $\{\eta(t)\}$ *in* $C(\mathbf{R}^+)$.

As a consequence, we have that the null distributions of test statistics $\widehat{M}_{c,a}^{\mathcal{I}}$ and $\widehat{J}_{c,a}^{\mathcal{I}}$ can be approximated with proposed bootstrapped procedure.

Percentage of rejected hypotheses, the simple hypothesis, $p = 0.1$

Alt.	$Exp(1)$	$W(1.4)$	$\Gamma(2)$	HN	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EW(1.5)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$\widehat{J}^{\mathcal{P}}_{c,1}$	5	74	96	39	16	9	97	45	58	75	85	67	76	100	37	80
$\widehat{J}^{\mathcal{P}}_{c,2}$	5	73	97	39	3	4	92	35	43	75	80	84	75	100	38	59
$\widehat{J}^{\mathcal{P}}_{c,5}$	4	70	97	35	1	2	79	23	25	71	72	93	75	100	40	37
$\widehat{J}^{\mathcal{D}}_{c,1}$	5	71	93	29	52	11	96	40	58	60	96	34	79	100	39	96
$\widehat{J}^{\mathcal{D}}_{c,2}$	5	72	97	33	18	6	94	36	48	66	92	65	82	100	40	82
$\widehat{J}^{\mathcal{D}}_{c,5}$	4	71	98	35	3	3	84	26	30	69	83	88	81	100	42	54
$\widehat{M}^{\mathcal{P}}_{c,1}$	5	74	96	39	26	11	97	47	61	75	89	65	76	100	36	85
$\widehat{M}^{\mathcal{P}}_{c,2}$	4	74	97	39	6	6	95	40	49	75	83	80	75	100	38	68
$\widehat{M}^{\mathcal{P}}_{c,5}$	4	71	98	37	1	3	85	28	31	73	76	91	75	100	39	42
$\widehat{M}^{\mathcal{D}}_{c,1}$	5	69	92	29	63	12	96	39	58	59	97	42	78	100	38	97
$\widehat{M}^{\mathcal{D}}_{c,2}$	5	71	97	32	28	8	95	37	53	65	95	67	82	100	40	88
$\widehat{M}^{\mathcal{D}}_{c,5}$	4	71	98	34	4	4	89	30	36	68	88	86	82	100	42	65
ω^2	5	15	100	8	100	100	100	98	100	15	95	70	100	100	13	100
A_{n3}	6	53	100	55	100	100	100	100	100	92	81	98	98	100	29	100
Q_n	5	65	96	42	99	100	100	97	100	79	81	42	68	99	35	51
Q_n^S	5	68	88	47	99	100	100	99	100	84	87	39	63	97	28	33
Δ_n	5	57	98	24	0	1	41	10	8	58	65	96	74	99	43	17

Alt.	$Exp(1)$	$W(1.4)$	$\Gamma(2)$	HN	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EW(1.5)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$\widehat{J}^{\mathcal{P}}_{c,1}$	5	69	96	34	6	10	93	39	51	71	85	58	71	99	31	61
$\widehat{J}^{\mathcal{P}}_{c,2}$	5	65	97	33	2	5	85	31	34	68	80	74	71	99	31	38
$\widehat{J}^{\mathcal{P}}_{c,5}$	5	60	97	28	1	5	67	20	21	62	72	85	70	99	32	20
$\widehat{J}^{\mathcal{D}}_{c,1}$	5	67	93	26	40	11	95	38	53	56	95	33	76	99	36	87
$\widehat{J}^{\mathcal{D}}_{c,2}$	5	68	97	29	9	8	90	33	42	61	92	56	78	100	36	67
$\widehat{J}^{\mathcal{D}}_{c,5}$	5	63	98	28	2	5	74	22	25	60	84	79	76	99	35	36
$\widehat{M}^{\mathcal{P}}_{c,1}$	5	69	96	34	12	12	94	41	55	71	88	56	70	99	31	69
$\widehat{M}^{\mathcal{P}}_{c,2}$	5	67	97	33	3	7	90	34	42	70	83	70	71	99	31	48
$\widehat{M}^{\mathcal{P}}_{c,5}$	5	61	97	30	1	5	74	23	26	64	75	82	71	99	32	26
$\widehat{M}^{\mathcal{D}}_{c,1}$	5	67	92	26	54	13	95	38	55	55	96	39	75	99	36	92
$\widehat{M}^{\mathcal{D}}_{c,2}$	5	67	96	28	18	9	91	35	46	61	94	58	78	100	36	76
$\widehat{M}^{\mathcal{D}}_{c,5}$	5	65	98	28	3	6	79	26	30	61	87	77	77	99	35	46
ω^2	6	13	100	8	100	100	100	93	100	14	93	69	100	100	12	100
A_{n3}	7	42	100	47	100	100	100	98	100	77	79	96	97	100	21	100
Q_n	5	55	94	35	98	100	100	92	100	68	78	41	61	98	32	46
Q_n^S	5	57	77	40	98	100	100	97	100	74	77	37	50	89	25	37
Δ_n	4	47	98	20	0	4	33	11	14	47	65	89	73	98	32	9

Alt.	$Exp(1)$	$W(1.4)$	$\Gamma(2)$	HN	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EW(1.5)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$\widehat{J}^{\mathcal{P}}_{c,1}$	5	62	94	28	3	9	86	32	45	59	80	46	71	99	25	37
$\widehat{J}^{\mathcal{P}}_{c,2}$	5	59	95	25	2	6	74	26	32	55	75	62	70	99	25	19
$\widehat{J}^{\mathcal{P}}_{c,5}$	5	53	96	22	1	6	55	19	22	47	69	73	71	98	24	8
$\widehat{J}^{\mathcal{D}}_{c,1}$	5	65	93	27	26	12	91	35	51	49	94	25	75	99	31	75
$\widehat{J}^{\mathcal{D}}_{c,2}$	5	64	96	26	6	9	82	28	39	52	89	46	77	99	30	46
$\widehat{J}^{\mathcal{D}}_{c,5}$	5	58	97	23	2	7	61	22	26	48	80	66	76	99	27	17
$\widehat{M}^{\mathcal{P}}_{c,1}$	5	63	94	29	6	10	88	34	49	60	82	45	71	99	25	46
$\widehat{M}^{\mathcal{P}}_{c,2}$	5	59	95	26	2	7	80	29	37	57	77	58	71	99	25	25
$\widehat{M}^{\mathcal{P}}_{c,5}$	5	56	96	23	1	6	62	21	25	50	71	70	71	99	24	10
$\widehat{M}^{\mathcal{D}}_{c,1}$	5	64	92	27	37	13	92	35	51	49	95	30	74	99	32	82
$\widehat{M}^{\mathcal{D}}_{c,2}$	5	64	96	26	11	10	84	30	42	52	92	47	77	99	30	57
$\widehat{M}^{\mathcal{D}}_{c,5}$	5	60	97	24	3	7	68	24	29	49	84	65	77	99	29	25
ω^2	4	13	100	7	100	100	98	82	98	11	92	67	100	100	11	100
A_{n3}	7	38	100	38	100	100	99	91	100	59	80	93	97	100	19	100
Q_n	5	48	92	28	96	98	100	82	97	50	66	30	54	95	26	48
Q_n^S	4	46	52	33	98	100	100	92	100	55	55	29	32	68	17	49
Δ_n	4	41	97	16	1	6	33	13	18	32	65	77	75	98	21	3

- for all considered censoring rates almost all tests are well calibrated:
  - for smaller a $J^{\mathcal{I}}_{c,a}$ are better calibrated.
  - $M^{\mathcal{D}}_{c,a}$, and $M^{\mathcal{P}}_{c,1}$ the level of significance is kept even for $p = 0.3$.
- power performance:
  - in most cases power decreases with the increase of initial censoring level
  - in some cases the censoring level doesn't have significant impact on test power
  - the ordering of tests shown to be not sensitive to the change of censoring rate
  - no test outperforms all competitors for all selected alternatives:
  - the powers of new tests are usually not much affected by choice of a.
  - Some differences can be seen in the case of decreasing-increasing failure rate alternatives such as CH(0.5)

CONCLUSION: In the case of testing simple hypothesis $J^{\mathcal{D}}_{c,1}$, $J^{\mathcal{P}}_{c,1}$, $M^{\mathcal{P}}_{c,1}$, $M^{\mathcal{D}}_{c,1}$, and A_{n3} deserve to be included in a battery of exponentiality tests for censored data

- The statistic can be made asymptotically scale free
- The bootstrap algorithm analogous to the one used in simple hypothesis case, is proposed. The only difference is in step 2. b., where we now generate a new sample from $\mathcal{E}\left(\frac{1}{\widehat{\mu}}\right)$, where

  $\widehat{\mu} = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} \delta_i}$ is MLE of μ
- The consistency of $\widehat{\mu}$ enables us to prove the consistency of bootstrap procedure

Alt.	$Exp(1)$	$W(1.4)$	$\Gamma(2)$	HN	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EW(1.5)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$\widehat{J}^{\mathcal{P}}_{c,1}$	5	77	94	45	91	30	100	64	83	79	82	56	66	99	40	98
$\widehat{J}^{\mathcal{P}}_{c,2}$	5	76	93	48	90	30	100	66	82	81	73	66	60	99	40	97
$\widehat{J}^{\mathcal{P}}_{c,5}$	4	73	90	47	88	29	100	65	81	81	63	78	50	98	40	97
$\widehat{J}^{\mathcal{D}}_{c,1}$	4	72	92	30	94	25	99	52	77	63	93	35	77	99	41	100
$\widehat{J}^{\mathcal{D}}_{c,2}$	5	75	94	36	92	27	100	59	82	72	88	54	70	99	42	99
$\widehat{J}^{\mathcal{D}}_{c,5}$	4	74	93	44	90	27	100	62	81	78	73	72	61	99	43	98
$\widehat{M}^{\mathcal{P}}_{c,1}$	5	77	93	44	92	29	100	63	83	78	86	56	68	99	39	99
$\widehat{M}^{\mathcal{P}}_{c,2}$	5	77	93	48	91	30	100	66	83	81	77	64	63	99	41	98
$\widehat{M}^{\mathcal{P}}_{c,5}$	4	74	91	47	89	29	100	66	82	82	66	74	53	98	39	97
$\widehat{M}^{\mathcal{D}}_{c,1}$	4	71	92	29	96	23	99	50	76	62	95	42	72	99	40	100
$\widehat{M}^{\mathcal{D}}_{c,2}$	5	74	93	35	93	26	100	58	81	71	90	54	70	99	41	99
$\widehat{M}^{\mathcal{D}}_{c,5}$	5	76	93	42	91	27	100	61	81	77	79	69	64	99	42	98
ω^2	5	69	87	40	93	25	99	58	79	77	76	87	51	97	39	98
A_{n3}	5	61	74	41	83	25	97	53	73	74	53	82	39	88	30	93
Q_n	5	68	91	26	89	15	97	44	67	58	85	20	64	98	32	95
Δ_n	4	66	81	42	87	25	99	60	79	79	43	87	35	91	40	96

$p = 0.2$ and $p = 0.3$

Alt.	$Exp(1)$	$W(1.4)$	$\Gamma(2)$	HN	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EW(1.5)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$\widehat{J}^{\mathcal{P}}_{c,1}$	5	69	89	39	84	26	99	52	73	73	80	44	64	97	32	92
$\widehat{J}^{\mathcal{P}}_{c,2}$	5	66	88	39	82	25	98	52	71	75	70	53	57	97	31	90
$\widehat{J}^{\mathcal{P}}_{c,5}$	5	62	83	35	80	23	96	49	67	73	57	64	49	93	30	88
$\widehat{J}^{\mathcal{D}}_{c,1}$	6	91	90	30	90	22	98	46	67	57	94	28	74	98	36	96
$\widehat{J}^{\mathcal{D}}_{c,2}$	6	91	91	35	87	23	98	49	69	66	87	43	70	98	36	94
$\widehat{J}^{\mathcal{D}}_{c,5}$	5	64	89	36	82	22	97	48	67	70	71	60	59	97	33	91
$\widehat{M}^{\mathcal{P}}_{c,1}$	5	69	89	38	86	26	99	52	73	72	82	44	64	97	32	93
$\widehat{M}^{\mathcal{P}}_{c,2}$	5	67	88	39	83	26	98	53	72	74	73	51	59	96	32	91
$\widehat{M}^{\mathcal{P}}_{c,5}$	5	63	84	37	81	24	97	50	69	74	61	62	51	94	30	89
$\widehat{M}^{\mathcal{D}}_{c,1}$	6	67	89	29	91	21	98	45	66	56	95	33	73	98	37	97
$\widehat{M}^{\mathcal{D}}_{c,2}$	6	68	91	34	88	22	98	48	69	64	90	44	71	98	36	95
$\widehat{M}^{\mathcal{D}}_{c,5}$	5	66	89	36	83	23	98	49	68	69	77	58	62	97	34	91
ω^2	5	57	78	33	91	21	94	43	63	65	70	78	48	93	31	96
A_{n3}	6	51	68	32	79	23	95	44	63	67	49	71	38	85	23	89
Q_n	5	60	84	27	86	15	96	37	57	54	80	22	59	96	30	90
Δ_n	4	52	68	29	78	19	93	43	59	65	38	71	34	81	30	86
$\widehat{J}^{\mathcal{P}}_{c,1}$	6	61	85	33	71	22	96	43	43	64	70	35	58	96	27	85
$\widehat{J}^{\mathcal{P}}_{c,2}$	5	57	82	32	67	20	94	40	56	63	60	43	50	94	26	82
$\widehat{J}^{\mathcal{P}}_{c,5}$	5	52	74	28	64	17	89	36	50	58	50	52	42	87	24	79
$\widehat{J}^{\mathcal{D}}_{c,1}$	5	64	89	28	80	20	96	39	62	53	91	24	72	98	34	93
$\widehat{J}^{\mathcal{D}}_{c,2}$	5	62	89	31	74	20	95	41	60	58	81	36	66	98	32	88
$\widehat{J}^{\mathcal{D}}_{c,5}$	5	56	84	29	68	18	92	37	52	59	62	51	54	94	27	83
$\widehat{M}^{\mathcal{P}}_{c,1}$	5	61	85	33	73	23	96	44	61	64	73	35	59	96	26	86
$\widehat{M}^{\mathcal{P}}_{c,2}$	5	59	83	33	69	21	95	41	58	63	63	41	52	94	26	84
$\widehat{M}^{\mathcal{P}}_{c,5}$	5	54	77	30	65	18	91	37	52	60	53	50	43	89	24	80
$\widehat{M}^{\mathcal{D}}_{c,1}$	5	62	89	27	81	20	96	39	62	52	92	28	71	98	34	94
$\widehat{M}^{\mathcal{D}}_{c,2}$	5	62	88	29	77	20	96	40	60	57	85	37	67	98	32	90
$\widehat{M}^{\mathcal{D}}_{c,5}$	5	58	85	29	70	18	93	38	54	59	68	48	57	95	29	85
ω^2	3	41	61	21	82	15	79	30	45	47	55	64	37	79	26	94
A_{n3}	6	48	64	29	73	20	89	38	52	60	48	59	35	81	20	86
Q_n	4	52	77	22	78	15	94	32	51	48	69	23	51	94	28	85
Δ_n	4	40	55	21	63	14	81	30	43	46	33	56	28	66	22	77

- most of considered tests are well calibrated
- the most powerful, with the exception of $LN(1.5)$ alternative, are proposed tests based on U-empirical Laplace transforms
- the power of proposed tests is not much affected by the change of censoring level

When $p = 0.1$ we recommend $J^{\mathcal{P}}_{c,1}, M^{\mathcal{P}}_{c,2}$, for $p = 0.2$ we recommend $J^{\mathcal{P}}_{c,1}, M^{\mathcal{D}}_{c,1}$, for larger censoring rates tests $J^{\mathcal{D}}_{c,1}$ and $M^{\mathcal{D}}_{c,1}$ are the optimal choice

To adapt, to impute or to amputate?

To adapt, to impute or to amputate?

- There are plenty missing data imputation algorithms, but most of them are for MAR data
- Balakrishnan et al. (2015) suggested to apply test on imputed data using very simple imputation procedure which keep test size at nominal level

Algorithm1 (proposed in Balakrishnan et al. 2015):

1. if $\delta_i = 0$ generate U_i from $[F_0(C_i), 1]$;
2. replace X_i with $F_0^{-1}(U_i)$, i.e. $X_i := F_0^{-1}(U_i)$, $\delta_i = 1$
3. repeat step 1 and step 2 for all sample elements
4. calculate test statistics based on modified sample

0. estimate $F(x)$ with $\widetilde{F}_n(x)$ smooth approximation of KM estimator $F_n(x)$;
1. if $\delta_i = 0$ generate U_i from $[\widetilde{F}_n(C_i), 1]$;
2. replace X_i with $\widetilde{F}_n^{-1}(U_i)$, i.e. $X_i := \widetilde{F}_0^{-1}(U_i)$, $\delta_i = 1$
3. repeat step 1 and step 2 for all sample elements
4. calculate test statistics based on modified sample

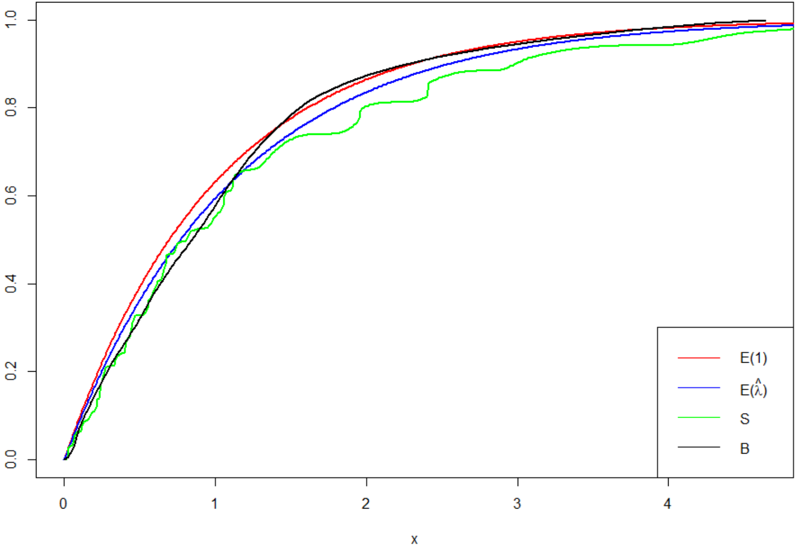
Let $X_{I(1)},...,X_{I(N)}$ uncensored observations ($N=\sum_{i=1}^{n}\delta_i$)

- monotone cubic spline based on $(X_{I(i)},\hat{S}_n(X_{I(i)}))$
- Bezier curve
  Kim, C. et al. (2003)
  N- the number of uncensored observations $X_{I(1)},...,X_{I(N)}$
  Bezier points $b_0=(0,1),\ b_i=(X_{I(i)},\hat{S}_n(X_{I(i)})),\ i=1,2...,N$
  and $b_{N+1}=(A_n,0)$, where $A_n=(1+\frac{1}{N})X_{I(N)}$
  Bezier curve is then

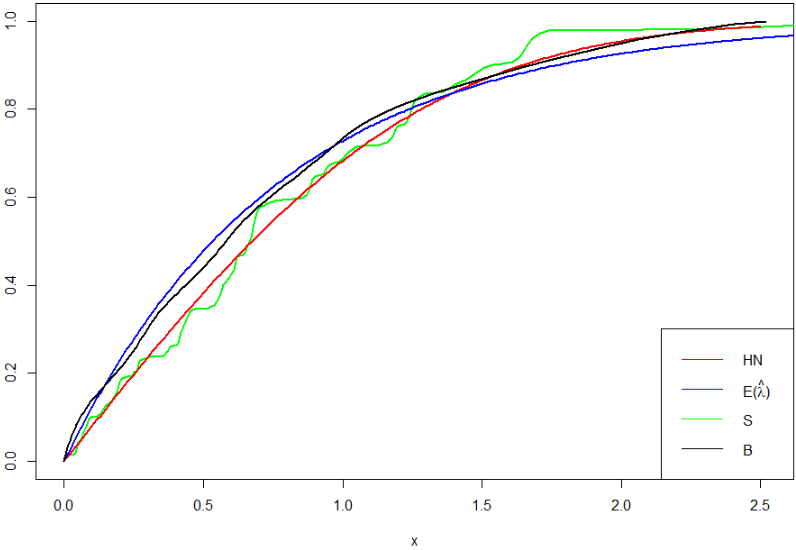
$$b(t)=(\hat{x}(t),\hat{y}(t))=\sum_{i=0}^{N+1}b_i\binom{N+1}{i}t^i(1-t)^{k-i}$$

$$\hat{S}_B(x)=\hat{y}(\hat{x}^{-1}(x))$$

$\mathcal{E}(\lambda)$



HN



Calculation of empirical powers, $n = 50$, $\alpha = 0.05$

- 4 approaches
- $p \in \{0.1, 0.2, 0.3\}$ percentage of censored data
- 15 test statistics
- 15 alternative distributions
- testing simple and composite hypothesis of exponentiality

Calibration of the new imputation algorithm (simple hypothesis, $p = 10\%$)

$100\alpha\%$	1	2	3	4	5	6
$\widehat{J}^{\mathcal{D}}_{c,1}$	1	2	4	5	6	7
$\widehat{J}^{\mathcal{D}}_{c,2}$	2	3	4	5	6	8
$\widehat{J}^{\mathcal{D}}_{c,5}$	2	3	4	6	7	8
$\widehat{J}^{\mathcal{P}}_{c,1}$	2	3	4	5	7	8
$\widehat{J}^{\mathcal{P}}_{c,2}$	2	3	5	6	7	8
$\widehat{J}^{\mathcal{P}}_{c,5}$	2	3	5	6	7	8
$\widehat{M}^{\mathcal{D}}_{c,1}$	1	2	4	5	6	7
$\widehat{M}^{\mathcal{D}}_{c,2}$	2	3	4	5	6	7
$\widehat{M}^{\mathcal{D}}_{c,5}$	2	3	4	6	7	8
$\widehat{M}^{\mathcal{P}}_{c,1}$	2	3	4	5	7	8
$\widehat{M}^{\mathcal{P}}_{c,2}$	2	3	5	6	7	8
$\widehat{M}^{\mathcal{P}}_{c,5}$	2	3	5	6	7	8
Q_n	2	3	4	6	7	8
Δ_n	2	3	5	6	7	8
ω^2	2	3	4	5	6	8

Empirical powers - simple hypothesis ($p = 0.1$)

Alt.	$E(1)$	$W(1.4)$	$\Gamma(2)$	HN	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$EV(1.5)$
$J^{\mathcal{D}}_{im1,1}$	5	68	95	24	88	4	87	51
$J^{\mathcal{D}}_{im2,1}$	6 (5)	74 (71)	94 (93)	32 (29)	58 (52)	13 (11)	98 (97)	62 (59)
$\widehat{J}^{\mathcal{D}}_{c,1}$	5	71	93	29	52	11	96	60
$J^{\mathcal{D}}_{am,1}$	5	67	92	25	43	8	95	51
$J^{\mathcal{P}}_{im1,1}$	5	72	98	33	69	3	83	67
$J^{\mathcal{P}}_{im2,1}$	6 (5)	81 (78)	97 (97)	47 (44)	25 (20)	14 (11)	99 (98)	82 (80)
$\widehat{J}^{\mathcal{P}}_{c,1}$	5	74	96	39	16	9	97	75
$J^{\mathcal{P}}_{am,1}$	4	73	96	36	11	7	96	70
$M^{\mathcal{D}}_{im1,1}$	5	66	94	24	92	4	89	50
$M^{\mathcal{D}}_{im2,1}$	6 (5)	73 (69)	93 (92)	30 (27)	68 (64)	14 (12)	98 (97)	61 (58)
$\widehat{M}^{\mathcal{D}}_{c,1}$	5	69	92	29	63	12	96	59
$M^{\mathcal{D}}_{am,1}$	5	66	91	24	58	9	95	50
$M^{\mathcal{P}}_{im1,1}$	5	72	98	33	77	4	88	67
$M^{\mathcal{P}}_{im2,1}$	7 (5)	81 (78)	97 (97)	46 (43)	37 (32)	17 (15)	99 (99)	81 (79)
$\widehat{M}^{\mathcal{P}}_{c,1}$	5	74	96	39	26	11	97	75
$M^{\mathcal{P}}_{am,1}$	4	73	95	36	22	10	97	70
Q_{im1}	5	67	97	40	97	98	100	72
Q_{im2}	7 (4)	75 (65)	97 (95)	54 (43)	100 (100)	100 (100)	100 (100)	89 (79)
Q_n	5	65	96	42	99	100	100	79
Q_{am}	5	74	93	54	100	100	100	84
Δ_{im1}	5	55	99	20	13	0	17	84
Δ_{im2}	7 (5)	70 (59)	99 (99)	34 (23)	1 (0)	0 (0)	55 (35)	90 (87)
Δ_n	5	57	98	24	0	1	41	58
Δ_{am}	3	55	98	19	0	0	33	51
ω^2_{im1}	5	17	100	8	100	100	100	13
ω^2_{im2}	6 (5)	20 (17)	100 (100)	12 (10)	100 (100)	100 (100)	100 (100)	23 (18)
ω^2	6	13	100	8	100	100	100	14
ω^2_{am}	8	11	100	20	100	100	100	24

Alt.	$LF(2)$	$LF(4)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$J^{\mathcal{D}}_{im1,1}$	27	36	99	21	82	100	40	99
$J^{\mathcal{D}}_{im2,1}$	43 (40)	61 (58)	96 (95)	37 (34)	79 (76)	100 (99)	44 (41)	96 (95)
$\widehat{J}^{\mathcal{D}}_{c,1}$	40	58	96	34	79	100	39	96
$J^{\mathcal{D}}_{am,1}$	34	51	93	28	78	99	40	93
$J^{\mathcal{P}}_{im1,1}$	29	33	95	46	81	100	39	94
$J^{\mathcal{P}}_{im2,1}$	53 (49)	66 (62)	93 (92)	66 (63)	77 (75)	100 (100)	43 (40)	87 (84)
$\widehat{J}^{\mathcal{P}}_{c,1}$	45	58	85	67	76	100	37	80
$J^{\mathcal{P}}_{am,1}$	40	53	87	57	75	100	37	79
$M^{\mathcal{D}}_{im1,1}$	28	39	99	23	80	99	39	99
$M^{\mathcal{D}}_{im2,1}$	43 (39)	62 (58)	96 (95)	45 (41)	78 (75)	100 (99)	43 (40)	97 (96)
$\widehat{M}^{\mathcal{D}}_{c,1}$	39	58	97	42	78	100	38	97
$M^{\mathcal{D}}_{am,1}$	34	52	93	33	76	99	38	95
$M^{\mathcal{P}}_{im1,1}$	32	39	95	42	80	100	40	96
$M^{\mathcal{P}}_{im2,1}$	55 (51)	70 (67)	93 (92)	64 (61)	77 (75)	100 (100)	44 (40)	91 (89)
$\widehat{M}^{\mathcal{P}}_{c,1}$	47	61	89	65	76	100	36	85
$M^{\mathcal{P}}_{am,1}$	43	58	89	54	75	100	37	86
Q_{im1}	91	99	90	34	75	99	35	43
Q_{im2}	99 (98)	100 (100)	88 (84)	44 (36)	74 (65)	99 (98)	39 (33)	59 (53)
Q_n	97	100	81	42	68	99	35	51
Q_{am}	99	100	91	39	64	98	25	77
Δ_{im1}	4	1	84	90	77	100	43	45
Δ_{im2}	12 (6)	7 (3)	90 (87)	94 (93)	75 (70)	100 (99)	46 (40)	28 (19)
Δ_n	10	8	65	96	74	99	43	17
Δ_{am}	5	2	45	96	70	99	41	100
ω^2_{im1}	91	100	67	100	100	100	13	100
ω^2_{im2}	98 (97)	100 (100)	75 (72)	100 (100)	100 (100)	100 (100)	16 (14)	100 (100)
ω^2	93	100	93	69	100	100	12	100
ω^2_{am}	100	100	21	46	98	100	22	100

Calibration of the new imputation algorithm (simple hypothesis, $p = 30\%$)

$100\alpha\%$	1	2	3	4	5	6
$\widehat{J}_{c,1}^{\mathcal{D}}$	2	4	5	7	8	10
$\widehat{J}_{c,2}^{\mathcal{D}}$	3	4	7	8	10	11
$\widehat{J}_{c,5}^{\mathcal{D}}$	3	6	8	10	11	13
$\widehat{J}_{c,1}^{\mathcal{P}}$	3	6	8	10	11	13
$\widehat{-}J_{c,2}^{\mathcal{P}}$	4	6	9	11	12	14
$\widehat{J}_{c,5}^{\mathcal{P}}$	2	7	9	11	13	15
$\widehat{M}_{c,1}^{\mathcal{D}}$	3	4	5	7	8	10
$\widehat{M}_{c,2}^{\mathcal{D}}$	3	4	7	8	9	11
$\widehat{M}_{c,5}^{\mathcal{D}}$	4	6	8	10	11	13
$\widehat{M}_{c,1}^{\mathcal{P}}$	4	6	8	10	11	13
$\widehat{M}_{c,2}^{\mathcal{P}}$	4	6	9	10	12	14
$\widehat{M}_{c,5}^{\mathcal{P}}$	4	7	9	11	13	14
Q_n	4	6	8	10	12	14
Δ_n	4	6	6	10	12	14
ω^2	3	5	6	8	9	10

Empirical powers - simple hypothesis ($p = 0.3$)

Alt.	$E(1)$	$W(1.4)$	$\Gamma(2)$	HN	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$EV(1.5)$
$J^{\mathcal{D}}_{im1,1}$	5	55	97	16	99	6	45	31
$J^{\mathcal{D}}_{im2,1}$	8 (5)	74 (67)	94 (90)	35 (28)	50 (41)	16 (11)	94 (92)	62 (54)
$\widehat{J}^{\mathcal{D}}_{c,1}$	5	65	93	27	26	12	91	49
$J^{\mathcal{D}}_{am,1}$	3	53	88	15	10	3	81	31
$J^{\mathcal{P}}_{im1,1}$	5	55	99	18	97	7	29	38
$J^{\mathcal{P}}_{im2,1}$	10 (4)	81 (71)	97 (94)	53 (40)	25 (13)	18 (9)	94 (90)	80 (60)
$\widehat{J}^{\mathcal{P}}_{c,1}$	5	62	94	28	3	9	86	59
$J^{\mathcal{P}}_{am,1}$	3	57	93	20	1	2	80	44
$M^{\mathcal{D}}_{im1,1}$	5	55	96	15	99	6	52	31
$M^{\mathcal{D}}_{im2,1}$	8 (5)	73 (66)	93 (90)	35 (27)	61 (51)	17 (12)	95 (93)	62 (53)
$\widehat{M}^{\mathcal{D}}_{c,1}$	5	64	92	27	37	13	92	49
$M^{\mathcal{D}}_{am,1}$	4	52	86	15	20	4	82	30
$M^{\mathcal{P}}_{im1,1}$	5	56	98	19	98	7	37	39
$M^{\mathcal{P}}_{im2,1}$	11 (4)	81 (63)	97 (90)	52 (31)	34 (10)	22 (7)	96 (88)	80 (61)
$\widehat{M}^{\mathcal{P}}_{c,1}$	5	63	94	29	6	10	88	60
$M^{\mathcal{P}}_{am,1}$	3	58	92	21	2	3	85	44
Q_{im1}	5	49	97	22	51	55	84	40
Q_{im2}	12 (4)	75 (52)	97 (90)	59 (34)	98 (97)	98 (97)	100 (99)	84 (62)
Q_n	5	48	92	28	96	98	100	50
Q_{am}	17	79	81	66	100	100	100	84
Δ_{im1}	5	35	99	9	59	8	2	21
Δ_{im2}	12 (4)	72 (45)	99 (98)	40 (15)	4 (2)	1 (0)	52 (10)	67 (37)
Δ_n	4	41	97	16	1	6	33	32
Δ_{am}	1	24	95	4	0	0	4	15
ω^2_{im1}	5	15	100	5	100	99	85	7
ω^2_{im2}	9 (5)	26 (12)	100 (100)	20 (8)	100 (100)	100 (100)	100 (99)	32 (13)
ω^2	4	13	100	7	100	100	98	11
ω^2_{am}	42	27	99	60	100	100	100	58

Alt.	$LF(2)$	$LF(4)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$J^{\mathcal{D}}_{im1,1}$	9	8	98	6	86	100	33	99
$J^{\mathcal{D}}_{im2,1}$	46 (38)	60 (51)	95 (93)	27 (21)	80 (72)	99 (99)	40 (34)	90 (86)
$\widehat{J}^{\mathcal{D}}_{c,1}$	35	51	94	25	75	99	31	75
$J^{\mathcal{D}}_{am,1}$	19	28	93	14	73	98	28	69
$J^{\mathcal{P}}_{im1,1}$	8	5	94	11	87	100	28	97
$J^{\mathcal{P}}_{im2,1}$	56 (42)	65 (50)	92 (88)	43 (30)	80 (71)	99 (99)	35 (23)	73 (54)
$\widehat{J}^{\mathcal{P}}_{c,1}$	32	45	80	46	71	99	25	37
$J^{\mathcal{P}}_{am,1}$	20	27	88	30	73	99	22	35
$M^{\mathcal{D}}_{im1,1}$	10	11	98	6	85	99	32	99
$M^{\mathcal{D}}_{im2,1}$	46 (38)	61 (52)	96 (94)	32 (26)	79 (71)	99 (99)	40 (34)	93 (90)
$\widehat{M}^{\mathcal{D}}_{c,1}$	35	51	95	30	74	99	32	82
$M^{\mathcal{D}}_{am,1}$	19	31	94	15	72	98	27	78
$M^{\mathcal{P}}_{im1,1}$	9	7	95	9	86	100	27	98
$M^{\mathcal{P}}_{im2,1}$	58 (34)	68 (43)	93 (84)	41 (20)	80 (61)	99 (97)	36 (16)	80 (49)
$\widehat{M}^{\mathcal{P}}_{c,1}$	34	49	82	45	71	99	25	46
$M^{\mathcal{P}}_{am,1}$	23	33	90	26	73	99	22	47
Q_{im1}	47	65	89	16	77	99	25	20
Q_{im2}	95 (89)	99 (97)	88 (71)	40 (23)	74 (50)	99 (94)	38 (23)	66 (51)
Q_n	82	97	66	30	54	95	26	48
Q_{am}	100	100	91	19	55	94	100	97
Δ_{im1}	2	2	83	37	86	99	25	64
Δ_{im2}	17 (2)	9 (1)	90 (81)	60 (50)	84 (73)	99 (98)	31 (17)	20 (8)
Δ_n	13	18	65	77	75	98	21	3
Δ_{am}	0	0	47	75	63	97	16	1
ω^2_{im1}	61	91	93	51	99	100	12	100
ω^2_{im2}	96 (90)	100 (100)	95 (87)	74 (62)	100 (99)	100 (100)	21 (12)	100 (100)
ω^2	82	98	92	67	100	100	11	100
ω^2_{am}	100	100	21	8	69	95	62	100

$100\alpha\%$	1	2	3	4	5	6
$\widehat{J}^{\mathcal{D}}_{c,1}$	1	2	4	5	6	7
$\widehat{J}^{\mathcal{D}}_{c,2}$	1	2	4	5	6	7
$\widehat{J}^{\mathcal{D}}_{c,5}$	2	3	4	5	7	8
$\widehat{J}^{\mathcal{P}}_{c,1}$	1	2	4	5	7	8
$\widehat{J}^{\mathcal{P}}_{c,2}$	2	3	4	6	7	8
$\widehat{J}^{\mathcal{P}}_{c,5}$	2	3	4	6	7	8
$\widehat{M}^{\mathcal{D}}_{c,1}$	1	3	4	5	6	8
$\widehat{M}^{\mathcal{D}}_{c,2}$	1	2	4	5	6	7
$\widehat{M}^{\mathcal{D}}_{c,5}$	1	3	4	5	7	7
$\widehat{M}^{\mathcal{P}}_{c,1}$	1	3	4	5	7	8
$\widehat{M}^{\mathcal{P}}_{c,2}$	1	3	4	5	7	8
$\widehat{M}^{\mathcal{P}}_{c,5}$	2	3	4	6	7	8
Q_n	1	2	4	5	6	7
Δ_n	2	3	5	6	7	8
ω^2	2	3	5	6	7	9

Empirical powers - composite hypothesis ($p = 0.1$)

Alt.	$E(1)$	$W(1.4)$	$\Gamma(2)$	HN	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$EV(1.5)$
$J^{\mathcal{D}}_{im1,1}$	5	68	92	27	97	17	97	52
$J^{\mathcal{D}}_{im2,1}$	6 (5)	74 (71)	94 (93)	32 (29)	98 (97)	20 (18)	98 (98)	59 (56)
$\widehat{J}^{\mathcal{D}}_{c,1}$	4	72	92	30	94	25	99	63
$J^{\mathcal{D}}_{am,1}$	5	63	90	22	97	14	93	41
$J^{\mathcal{P}}_{im1,1}$	5	73	91	40	95	26	99	74
$J^{\mathcal{P}}_{im2,1}$	7 (5)	82 (79)	95 (93)	50 (47)	96 (95)	35 (31)	100 (100)	84 (82)
$\widehat{J}^{\mathcal{P}}_{c,1}$	5	77	94	45	91	30	100	79
$J^{\mathcal{P}}_{am,1}$	5	71	91	34	95	22	98	64
$M^{\mathcal{D}}_{im1,1}$	5	67	91	26	97	16	96	51
$M^{\mathcal{D}}_{im2,1}$	6 (5)	72 (68)	93 (92)	31 (28)	97 (97)	19 (17)	98 (98)	59 (55)
$\widehat{M}^{\mathcal{D}}_{c,1}$	4	71	92	29	96	23	99	62
$M^{\mathcal{D}}_{am,1}$	5	62	89	21	96	13	93	40
$M^{\mathcal{P}}_{im1,1}$	5	73	92	39	95	25	99	73
$M^{\mathcal{P}}_{im2,1}$	7 (5)	81 (79)	95 (94)	49 (45)	96 (95)	33 (30)	100 (100)	83 (81)
$\widehat{M}^{\mathcal{P}}_{c,1}$	5	77	93	44	92	29	100	78
$M^{\mathcal{P}}_{am,1}$	5	70	91	33	95	21	98	62
Q_{im1}	5	65	89	27	92	17	97	55
Q_{im2}	6 (5)	70 (66)	92 (89)	32 (28)	92 (90)	20 (17)	98 (98)	63 (58)
Q_n	5	68	91	26	89	15	97	58
Q_{am}	5	66	90	25	88	15	97	53
Δ_{im1}	5	63	79	39	90	27	96	74
Δ_{im2}	7 (5)	78 (72)	89 (85)	55 (47)	90 (88)	40 (33)	100	90 (86)
Δ_n	4	66	81	42	87	25	99	79
Δ_{am}	5	68	85	37	92	23	99	73
ω^2_{im1}	5	64	84	37	42	25	98	73
ω^2_{im2}	7 (5)	75 (68)	50 (42)	90 (86)	94 (91)	35 (28)	100 (99)	86 (81)
ω^2	5	69	87	40	93	25	99	77
ω^2_{am}	5	64	85	34	93	21	98	66

Alt.	$LF(2)$	$LF(4)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$J^{\mathcal{D}}_{im1,1}$	40	59	86	48	68	99	41	99
$J^{\mathcal{D}}_{im2,1}$	46 (43)	66 (63)	90 (89)	57 (54)	71 (68)	99 (99)	44 (41)	100 (100)
$\widehat{J}^{\mathcal{D}}_{c,1}$	52	77	93	35	73	99	41	100
$J^{\mathcal{D}}_{am,1}$	32	50	94	32	71	99	37	99
$J^{\mathcal{P}}_{im1,1}$	56	75	47	60	60	98	39	99
$J^{\mathcal{P}}_{im2,1}$	66 (63)	84 (82)	85 (84)	69 (65)	65 (62)	99 (98)	42 (37)	99 (99)
$\widehat{J}^{\mathcal{P}}_{c,1}$	64	83	82	56	66	99	40	98
$J^{\mathcal{P}}_{am,1}$	48	69	91	44	65	99	37	99
$M^{\mathcal{D}}_{im1,1}$	39	58	73	46	69	99	40	99
$M^{\mathcal{D}}_{im2,1}$	45 (42)	65 (62)	91 (90)	57 (53)	72 (68)	99 (99)	42 (39)	100 (100)
$\widehat{M}^{\mathcal{D}}_{c,1}$	50	76	95	42	72	99	40	100
$M^{\mathcal{D}}_{am,1}$	31	49	94	31	71	99	35	99
$M^{\mathcal{P}}_{im1,1}$	55	74	51	57	62	98	38	99
$M^{\mathcal{P}}_{im2,1}$	65 (62)	83 (81)	86 (85)	66 (62)	67 (63)	99 (99)	42 (37)	99 (99)
$\widehat{M}^{\mathcal{P}}_{c,1}$	63	83	86	56	68	99	39	99
$M^{\mathcal{P}}_{am,1}$	47	68	92	40	67	99	34	99
Q_{im1}	41	61	56	21	62	98	34	96
Q_{im2}	47 (43)	67 (63)	84 (82)	25 (22)	66 (61)	99 (98)	36 (33)	95 (94)
Q_n	44	67	85	20	64	98	32	95
Q_{am}	39	59	91	11	67	99	29	92
Δ_{im1}	53	72	20	80	36	90	40	97
Δ_{im2}	70 (63)	87 (83)	77 (74)	83 (80)	45 (39)	96 (94)	42 (36)	97 (95)
Δ_n	60	79	43	87	35	91	40	96
Δ_{am}	52	72	76	80	46	96	41	97
ω^2_{im1}	52	71	38	78	48	95	36	98
ω^2_{im2}	65 (58)	83 (78)	86 (82)	83 (80)	55 (47)	98 (96)	41 (36)	98 (98)
ω^2	58	79	76	87	51	97	39	98
ω^2_{am}	47	67	85	77	55	97	37	98

$100\alpha\%$	1	2	3	4	5	6
$\widehat{J}^{\mathcal{D}}_{c,1}$	2	4	5	7	8	9
$\widehat{J}^{\mathcal{D}}_{c,2}$	3	5	7	8	10	11
$\widehat{J}^{\mathcal{D}}_{c,5}$	4	6	8	10	12	14
$\widehat{J}^{\mathcal{P}}_{c,1}$	4	6	8	10	12	13
$\widehat{J}^{\mathcal{P}}_{c,2}$	5	7	9	11	13	15
$\widehat{J}^{\mathcal{P}}_{c,5}$	5	8	10	12	14	17
$\widehat{M}^{\mathcal{D}}_{c,1}$	2	4	5	7	8	10
$\widehat{M}^{\mathcal{D}}_{c,2}$	3	5	6	8	9	11
$\widehat{M}^{\mathcal{D}}_{c,5}$	4	6	8	10	11	13
$\widehat{M}^{\mathcal{P}}_{c,1}$	4	6	8	10	11	13
$\widehat{M}^{\mathcal{P}}_{c,2}$	5	7	9	11	13	14
$\widehat{M}^{\mathcal{P}}_{c,5}$	5	8	10	12	14	15
Q_n	2	4	6	7	9	10
Δ_n	4	8	10	12	14	16
ω^2	5	8	11	13	16	18

Alt.	$E(1)$	$W(1.4)$	$\Gamma(2)$	HN	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$EV(1.5)$
$J^{\mathcal{D}}_{im1,1}$	5	54	81	18	94	12	87	36
$J^{\mathcal{D}}_{im2,1}$	8 (5)	75 (67)	93 (90)	35 (28)	95 (94)	23 (17)	97 (96)	60 (52)
$\widehat{J}^{\mathcal{D}}_{c,1}$	5	64	89	28	80	20	96	53
$J^{\mathcal{D}}_{am,1}$	5	56	87	17	95	11	88	32
$J^{\mathcal{P}}_{im1,1}$	5	53	74	24	88	16	87	46
$J^{\mathcal{P}}_{im2,1}$	12 (4)	83 (67)	93 (84)	55 (37)	91 (79)	41 (24)	99 (97)	82 (68)
$\widehat{J}^{\mathcal{P}}_{c,1}$	6	61	85	33	71	22	96	64
$J^{\mathcal{P}}_{am,1}$	5	64	89	26	93	17	96	51
$M^{\mathcal{D}}_{im1,1}$	5	53	82	18	94	12	87	35
$M^{\mathcal{D}}_{im2,1}$	8 (5)	74 (66)	93 (89)	34 (27)	96 (94)	23 (17)	97 (96)	60 (52)
$\widehat{M}^{\mathcal{D}}_{c,1}$	5	62	88	27	81	20	96	52
$M^{\mathcal{D}}_{am,1}$	5	54	86	16	95	10	88	31
$M^{\mathcal{P}}_{im1,1}$	5	54	75	24	89	16	88	46
$M^{\mathcal{P}}_{im2,1}$	11 (4)	82 (67)	93 (84)	54 (36)	91 (80)	40 (24)	99 (87)	82 (67)
$\widehat{M}^{\mathcal{P}}_{c,1}$	5	61	85	33	73	23	96	64
$M^{\mathcal{P}}_{am,1}$	5	63	89	26	93	17	95	50
Q_{im1}	5	52	77	19	87	12	88	39
Q_{im2}	9 (4)	73 (61)	91 (83)	36 (26)	91 (84)	24 (15)	98 (95)	64 (52)
Q_n	4	52	77	22	78	15	94	48
Q_{am}	5	53	83	16	79	11	89	33
Δ_{im1}	4	37	50	19	78	13	68	37
Δ_{im2}	14 (4)	81 (62)	88 (74)	61 (38)	78 (61)	48 (25)	97 (95)	86 (71)
Δ_n	4	40	55	21	63	14	81	46
Δ_{am}	5	60	83	28	91	18	97	58
ω^2_{im1}	5	44	64	21	83	14	82	43
ω^2_{im2}	16 (5)	80 (57)	90 (74)	58 (33)	90 (78)	45 (22)	97 (95)	85 (66)
ω^2	3	41	61	21	82	15	79	47
ω^2_{am}	5	57	82	25	91	16	94	52

Alt.	$LF(2)$	$LF(4)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$J^{\mathcal{D}}_{im1,1}$	29	44	69	21	59	95	32	98
$J^{\mathcal{D}}_{im2,1}$	48 (41)	67 (59)	90 (87)	39 (33)	74 (67)	99 (98)	40 (33)	99 (98)
$\widehat{J}^{\mathcal{D}}_{c,1}$	39	62	91	24	72	98	34	93
$J^{\mathcal{D}}_{am,1}$	26	42	94	20	69	98	33	99
$J^{\mathcal{P}}_{im1,1}$	36	51	47	26	48	88	27	95
$J^{\mathcal{P}}_{im2,1}$	68 (51)	84 (70)	85 (77)	42 (23)	71 (53)	98 (94)	34 (15)	96 (91)
$\widehat{J}^{\mathcal{P}}_{c,1}$	43	60	70	35	58	96	27	85
$J^{\mathcal{P}}_{am,1}$	39	59	90	28	66	98	30	98
$M^{\mathcal{D}}_{im1,1}$	28	44	74	21	60	95	32	98
$M^{\mathcal{D}}_{im2,1}$	48 (40)	67 (59)	91 (88)	40 (33)	74 (66)	99 (98)	40 (33)	99 (98)
$\widehat{M}^{\mathcal{D}}_{c,1}$	39	62	92	28	71	98	34	94
$M^{\mathcal{D}}_{am,1}$	24	41	94	18	69	97	32	99
$M^{\mathcal{P}}_{im1,1}$	36	52	52	25	50	90	26	96
$M^{\mathcal{P}}_{im2,1}$	67 (51)	83 (69)	86 (78)	41 (22)	71 (54)	98 (95)	34 (16)	97 (92)
$\widehat{M}^{\mathcal{P}}_{c,1}$	44	61	73	35	59	96	26	86
$M^{\mathcal{P}}_{am,1}$	39	58	91	25	68	98	29	98
Q_{im1}	30	46	58	18	52	92	28	94
Q_{im2}	51 (39)	69 (58)	85 (80)	37 (26)	68 (56)	98 (95)	37 (26)	95 (91)
Q_n	32	51	69	23	51	94	28	85
Q_{am}	26	43	91	6	64	97	24	85
Δ_{im1}	27	38	21	33	28	63	24	89
Δ_{im2}	73 (51)	86 (69)	77 (66)	41 (26)	60 (39)	93 (86)	28 (14)	88 (76)
Δ_n	30	43	33	56	28	66	22	77
Δ_{am}	41	62	75	67	49	95	38	97
ω^2_{im1}	31	45	41	35	38	83	24	93
ω^2_{im2}	70 (47)	84 (64)	87 (71)	56 (40)	66 (41)	97 (89)	39 (21)	90
ω^2	30	45	55	64	37	79	26	94
ω^2_{am}	38	58	85	64	56	95	33	97

- Algorithm 2, when appropriately calibrated is preferable imputation procedure
- For larger censoring rate it might not be a good choice if the calibration is not done!
- Imputation is computationally much more convenient for usage in practice (for larger sample sizes)
- Amputation works for testing composite hypothesis (for some test statistics)

- Explore quality of imputation algorithms for testing goodness-of-fit with other classes of null distributions as well as for two sample tests;
- Use multiple imputation procedure;
- Consider model adaptation for other censorship schemes;
- Adapt imputation procedures to other censorship schemes;

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