On Spatial Distribution of the Particle Field for Branching Random Walks

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Outline

Introduction. Branching Random Walks

Branching random walks are extremely useful in the study of stochastic systems with **birth, death** and **migration** of their elements.

The principal attention will be paid to the properties of branching random walks on multidimensional lattice $\mathbf{Z}^d$, $d \in \mathbf{N}$.

We will be mainly interested in the problems related to the limiting behavior of branching random walks such as

- existence of phase transitions under change of various parameters,
- the properties of the limiting distribution of the particle population,
- existence and the shape of the propagating fronts of particles,
- etc.

The answer to these and other questions heavily depends on numerous factors which affect the properties of a branching random walk.

We describe, how the properties of a branching walk depend on

- the homogeneity or non homogeneity of the branching media,
- the number and mutual disposition of the branching sources,
- the symmetry or non symmetry of the branching walk,
- the finiteness or infiniteness of the variance of jumps.

We present also some results of computer simulation of branching random walks and discuss how they may be applied to numerical estimation of various characteristics describing the properties of the phase transitions.

Introduction. The method of moments

It is shown how **the growth** of the limiting **moments** of the number of particles at each point of $\mathbf{Z}^d$, $d \in \mathbf{N}$, corresponds to the limiting structure of the particle field under various assumptions on intensities of generation and transport of particles.

The key question is in which cases the boundedness (in time) of normalized moments of the number of particles guarantees the uniqueness of the probability distribution and, as a consequence, convergence in distribution to a certain limiting random variable.

Limit theorems on the behavior of branching random walks in homogeneous and inhomogeneous media can be proved using by **the method of moments**.

Random walk

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- homogeneity: $a(x,y) = a(0,y-x) = a(y-x)$;

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- symmetry: $a(x,y) = a(y,x)$;
- homogeneity: $a(x,y) = a(0,y-x) = a(y-x)$;
- $a(x) \geq 0$ for $x \neq 0$, $-\infty < a(0) < 0$, $\sum_{x\in\mathbf{Z}^d} a(x) = 0$;

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- symmetry: $a(x,y) = a(y,x)$;
- homogeneity: $a(x,y) = a(0, y-x) = a(y-x)$;
- $a(x) \geq 0$ for $x \neq 0$, $-\infty < a(0) < 0$, $\sum\limits_{x\in \mathbf{Z}^d} a(x) = 0$;
- irreducibility: every point $y \in Z^d$ is reachable, i.e., for every $z \in \mathbf{Z^d}$ there exists a set of vectors $z_1, z_2, \ldots, z_k \in \mathbf{Z^d}$ such that $z = \sum_{i=1}^k z_i$ and $a(z_i) \neq 0$ for $i = 1, 2, \ldots, k$.

The transition probability

The random walk transition probability $p(t, x, y)$ satisfies the system of differential-difference equations (Kolmogorov's backward equations)

$$\frac{\partial p}{\partial t} = \mathcal{A}p, \quad p(0, x, y) = \delta_y(x),$$

where the (linear) operator $\mathcal{A}$ (the generator of the random walk) acts with respect to variable x as follows:

$$\mathcal{A}p(t, x, y) := \sum_{x' \in Z^d} a(x, x')p(t, x', y).$$

The Green Function

The Green function $G_\lambda(t,x,y)$ for $p(t,x,y)$ is defined as follows:

$$G_\lambda(t,x,y) := \int_0^t e^{-\lambda u} p(u,x,y)\, du, \quad \lambda \geq 0.$$

Analysis of random walks depends on whether the value of $G_0 = G_0(0,0) = \lim_{t\to\infty} G_0(t,0,0) = \int_0^\infty p(u,0,0)\, du$ is **finite** or **infinite**.

Recall that a random walk is called *transient* if $G_0 < \infty$ and *recurrent* if $G_0 = \infty$.

Assumptions on Variance of Random Walk Jumps

In what follows, we will assume that the function $a(\cdot)$ satisfy one of the following two different conditions:

1. either $\sum_z |z|^2 a(z) < \infty$, where $|z|$ is Euclidean norm of a vector z;
2. or $a(z) \sim \frac{H\left(\frac{z}{|z|}\right)}{|z|^{d+\alpha}}$, $\alpha \in (0,2)$, where $H(\cdot)$ is continuous positive and symmetric on the sphere $\mathbf{S}^{d-1} = \{z \in \mathbf{R}^d : |z| = 1\}$ function.

Under **Assumption 1**: $G_0 = \infty$ for $d = 1, 2$ and $G_0 < \infty$ for $d \geq 3$.

Under **Assumption 2**: $\sum_z |z|^2 a(z) = \infty$ which implies infinite variance of jumps. In this case $G_0 = \infty$ for $d = 1$ and $\alpha \in [1, 2)$, while $G_0 < \infty$ for $d = 1$ and $\alpha \in (0, 1)$ or $d \geq 2$ and $\alpha \in (0, 2)$.

Transition probabilities under Assumption 1

Theorem

For all $x, y \in \mathbf{Z}^d$, $d \geq 1$ *transition probabilities have the following form*

$$p(t, x, y) \sim \frac{\gamma_d}{t^{\frac{d}{2}}} \quad as \quad t \to \infty,$$

where $\gamma_d = \left((2\pi)^d D_d\right)^{-1/2}$, $D_d = |\det \phi''_{\theta\theta}(0)|$, $\phi(\theta) = \sum_x a(x) e^{i(x,\theta)}$.

Lemma

For $t \to \infty$ *we get*

$$G(t) := G_0(t, 0, 0) \sim \begin{cases} 2\,\gamma_1 \sqrt{t} & for \quad d = 1, \\ \gamma_2 \ln t & for \quad d = 2, \\ C_d < \infty & for \quad d \geq 3. \end{cases}$$

Transition probabilities under Assumption 2

Theorem

For all $x, y \in \mathbf{Z}^d$, $d \in \mathbf{N}$, and $0 < \alpha < 2$, transition probabilities have the following form

$$p(t,x,y) \sim \frac{h_{\alpha,d}}{t^{\frac{d}{\alpha}}} \quad \textit{as} \quad t \to \infty,$$

where $h_{\alpha,d} > 0$ is some constant.

Lemma

For $t \to \infty$ we get

$$G(t) := G_0(t,0,0) \sim \begin{cases} h_{\alpha,1}\frac{\alpha}{\alpha-1}t^{1-\frac{1}{\alpha}} & \textit{for} \quad d=1,\ 1<\alpha<2; \\ h_{1,1}\ln t & \textit{for} \quad d=1,\ \alpha=1; \\ h_{\alpha,d}<\infty & \textit{for} \quad d=1,\ 0<\alpha<1 \textit{ and } d\geq 2. \end{cases}$$

The sojourn time of the random walk at the lattice point

Let

$$\xi_t := \int_0^t \mathbf{I}_{\{X(s)\in\mathbf{0}\}}\, ds.$$

be the sojourn time of a trajectory of the random walk $X(t)$ at the origin during the time interval $(0, t]$.

Random walks with a finite variance of jumps

Theorem

*Let **Assumption 1** be valid and* $x \in \mathbf{R}_+$.

If $d = 1$ *(the recurrent case) then*

$$\lim_{t\to\infty} \mathbf{P}\left[\frac{2\xi_t}{\sqrt{\pi}\,G(t)} \leq x\right] = \frac{1}{\sqrt{\pi}}\int_0^x e^{-\frac{t^2}{4}}\,dt.$$

If $d = 2$ *(the recurrent case) then*

$$\lim_{t\to\infty} \mathbf{P}\left[\frac{\xi_t}{G(t)} \leq x\right] = 1 - e^{-x}.$$

If $d \geq 3$ *(transient cases) then*

$$\lim_{t\to\infty} \mathbf{P}\left[\frac{\xi_t}{G(t)} \leq x\right] = \begin{cases} 1 & \text{for} \quad x \geq 1, \\ 0 & \text{for} \quad 0 \leq x < 1. \end{cases}$$

Random walks under Assumption 2: case of infinite variance

Theorem

*Let **Assumption 2** be valid and $x \in \mathbf{R}_+$.*

If $d = 1$ and $1 < \alpha < 2$ (the recurrent case) then

$$\lim_{t\to\infty} \mathbf{P}\left[\frac{\xi_t}{\Gamma(2-1/\alpha)G(t)} \leq x\right] = \mathbf{P}\left(\zeta_{1-\frac{1}{\alpha}} \leq x\right),$$

where r.v. $\zeta_{1-\frac{1}{\alpha}}$ has the Mittag-Leffler distribution where

$$\mathbf{P}\left(\zeta_{1-\frac{1}{\alpha}} \leq x\right) = \frac{1}{\pi\theta}\int_0^x \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j!} \sin\left(\pi\theta j\right)\Gamma(\theta j+1)y^{j-1}\,ds.$$

If $d = 1$ and $\alpha = 1$ (the recurrent case) then

$$\lim_{t\to\infty} \mathbf{P}\left[\frac{\xi_t}{G^{(}t)} \leq x\right] = 1 - e^{-x}.$$

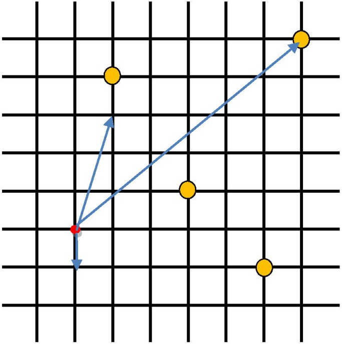
If $d \geq 1$, $0 < \alpha < 1$, or $d \geq 2$ (transient cases) then

$$\lim_{t\to\infty} \mathbf{P}\left[\frac{\xi_t}{G(t)} \leq x\right] = \begin{cases} 1 & \textit{for} \quad x \geq 1, \\ 0 & \textit{for} \quad 0 \leq x < 1. \end{cases}$$

Informal Description of BRWs on $\mathbf{Z}^d$ with a few branching sources

- An initial distribution of particles is given (e.g., the population of individuals is initiated at time $t = 0$ by a single particle at a point $x \in \mathbf{Z}^\mathbf{d}$).
- Being outside of the sources the particle performs a continuous time random walk on $\mathbf{Z}^\mathbf{d}$ until reaching one of the sources.
- At a source it spends an exponentially distributed time and then either jumps to a point $y \in \mathbf{Z}^\mathbf{d}$ (distinct from the source) or dies producing just before the death a random number of offsprings.
- The newborn particles behave independently and stochastically in the same way as the parent individual.

Example. BRW on $\mathbf{Z}^2$ with four sources of branching



Branching process at the source

The branching mechanism at the source is governed by the infinitesimal generating function

$$f(u) := \sum_{n=0}^{\infty} b_n u^n, \quad 0 \le u \le 1,$$

where

$$b_1 < 0, \quad b_n \ge 0, \quad n \ne 1, \quad \sum_n b_n = 0.$$

Suppose that

$$\beta_r := f^{(r)}(1) < \infty, \quad r \in \mathbf{N}, \qquad \beta := \beta_1.$$

Objects of study

Let $\mu_t(y)$ be the number of particles at the point $y \in \mathbf{Z^d}$ at the instant t, then

$$\mu_t := \sum_y \mu_t(y)$$

be the total number of particles (the population size) at the instant t and

$$\begin{aligned} m_n(t,x,y) &:= E_x \mu_t^n(y), \\ m_n(t,x) &:= E_x \mu_t^n, \quad n \in \mathbf{N}, \end{aligned}$$

where E_x denotes the mathematical expectation under the condition $\mu_0(\cdot) = \delta_x(\cdot)$.

The Main Equations

The moments m_n satisfy the chain of evolution equations:

$$\frac{\partial m_n}{\partial t} = \mathcal{H}_\beta m_n + \delta_0(x) g_n(m_1, \ldots, m_{n-1}),$$

with the initial conditions $m_n(0,x,y) = \delta_y(x)$, $m_n(0,x) \equiv 1$, respectively. Here

$$g_n(m_1, \ldots, m_{n-1}) := \sum_{r=2}^{n} \frac{\beta_r}{r!} \sum_{(i_1,\ldots,i_r)} \frac{n!}{i_1! \ldots i_r!} m_{i_1} \ldots m_{i_r},$$

where the second sum is taken over the integer r-tuples with $i_1, \ldots, i_r > 0$, $i_1 + \cdots + i_r = n$.

For $n = 1$, we have $g_1 \equiv 0$, so that $\partial m_1 / \partial t = \mathcal{H}_\beta m_1$.

The critical point

The operator $\mathcal{A}$ in $\ell^2(\mathbf{Z^d})$ has only the essential spectrum

$$\sigma(\mathcal{A}) = [\min_\theta \phi(\theta), 0], \quad \phi(\theta) := \sum_x a(x) e^{i(x,\theta)}, \quad \theta \in [-\pi, \pi]^d,$$

which coincides with the essential spectrum of the operator

$$\mathcal{H}_\beta = \mathcal{A} + \beta\delta_0(x).$$

Furthermore, for $\beta > \beta_c$ the operator $\mathcal{H}$ has the unique eigenvalue $\lambda_0 > 0$, which is a unique root of the equation

$$\beta G_\lambda(0,0) = 1,$$

where $G_\lambda(0,0) := \lim_{t\to\infty} G_\lambda(t,0,0) = \int\limits_0^t e^{-\lambda u} p(u,0,0)\, du$ for $\lambda \geq 0$.

The critical point

Put

$$\beta_c := 1/G_0(0,0).$$

Under **Assumption 1**: $G_0 = \infty$ for $d = 1, 2$ and $G_0 < \infty$ for $d \geq 3$. Therefore

$\beta_c = 0$ for $d = 1, 2$,

$\beta_c > 0$ for $d \geq 3$.

Under **Assumption 2**: $G_0 = \infty$ for $d = 1$ and $\alpha \in [1, 2)$. $G_0 < \infty$ for $d = 1$ and $\alpha \in (0, 1)$ or $d \geq 2$ and $\alpha \in (0, 2)$. Therefore

$\beta_c = 0$ for $d = 1$ and $\alpha \in [1, 2)$,

$\beta_c > 0$ for $d = 1$ and $\alpha \in (0, 1)$ or $d \geq 2$ and $\alpha \in (0, 2)$.

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The point $\beta_{\rm c}$ is critical since the asymptotic behavior of the process is essentially different for

- $\beta > \beta_c$,
- $\beta = \beta_c$,
- $\beta < \beta_c$.

Asymptotics of the moments under Assumption 1

For $t \to \infty$, the moments m_n have the following asymptotics:

$$m_n(t,x,y) \sim C_n^d(x,y)u_n(t), \quad m_n(t,x) \sim C_n^d(x)v_n(t),$$

where $C_n^d(x,y)$, $C_n^d(x)$ are the positive constants, and

If $\beta > \beta_c$ then $u_n(t) = v_n(t) = e^{n\lambda_0 t}$ for all d.

If $\beta = \beta_c$ then

$$\begin{array}{lll}
u_n(t) = t^{-1/2}(\ln t)^{n-1}, & v_n(t) = t^{(n-1)/2} & \textit{for} \quad d = 1; \\
u_n(t) = t^{-1}, & v_n(t) = (\ln t)^{n-1} & \textit{for} \quad d = 2; \\
u_n(t) = t^{-1/2}(\ln t)^{n-1}, & v_n = t^{n-1/2} & \textit{for} \quad d = 3; \\
u_n(t) = t^{n-1}(\ln t)^{1-2n}, & v_n = t^{2n-1}(\ln t)^{1-2n} & \textit{for} \quad d = 4; \\
u_n(t) = t^{n-1}, & v_n(t) = t^{2n-1} & \textit{for} \quad d \geq 5.
\end{array}$$

If $\beta < \beta_c$ then

$$\begin{array}{lll}
u_n(t) = t^{-3/2}, & v_n(t) = t^{-1/2} & \textit{for} \quad d = 1; \\
u_n(t) = (t\ln^2 t)^{-1}, & v_n(t) = (\ln t)^{-1} & \textit{for} \quad d = 2; \\
u_n(t) = t^{-d/2}, & v_n(t) \equiv 1 & \textit{for} \quad d \geq 3.
\end{array}$$

Asymptotics of the moments under Assumption 2 (Rytova & Y., 2019)

For $t \to \infty$ the moments m_n have the following asymptotics:

$$m_n(t,x,y) \sim C_n(x,y)u_n(t), \quad m_n(t,x) \sim C_n(x)v_n(t),$$

where

$u_n(t) = e^{n\lambda_0 t}$,	$v_n(t) = e^{n\lambda_0 t}$	if $\beta > \beta_c$, $d/\alpha \in (1/2, \infty)$;
$u_n(t) = t^{-1/\alpha}$,	$v_n(t) = t^{(1-1/\alpha)(n-1)}$	if $\beta = \beta_c$, $d/\alpha \in (1/2, 1)$;
$u_n(t) = t^{-1}$,	$v_n(t) = (\ln t)^{n-1}$	if $\beta = \beta_c$, $d/\alpha = 1$;
$u_n(t) = t^{d/\alpha-2}$,	$v_n(t) = t^{(d/\alpha-1)(2n-1)}$	if $\beta = \beta_c$, $d/\alpha \in (1, 3/2)$;
$u_n(t) = t^{-1/2}(\ln t)^{n-1}$,	$v_n(t) = t^{n-1/2}$	if $\beta = \beta_c$, $d/\alpha = 3/2$;
$u_n(t) = t^{(d/\alpha-2)(2n-1)+n-1}$,	$v_n(t) = t^{(d/\alpha-1)(2n-1)}$	if $\beta = \beta_c$, $d/\alpha \in (3/2, 2)$;
$u_n(t) = t^{n-1}(\ln t)^{1-2n}$,	$v_n(t) = t^{2n-1}(\ln t)^{1-2n}$	if $\beta = \beta_c$, $d/\alpha = 2$;
$u_n(t) = t^{n-1}$,	$v_n(t) = t^{2n-1}$	if $\beta = \beta_c$, $d/\alpha \in (2, \infty)$;
$u_n(t) = t^{1/\alpha-2}$,	$v_n(t) = t^{1/\alpha-1}$	if $\beta < \beta_c$, $d/\alpha \in (1/2, 1)$;
$u_n(t) = t^{-1}\ln^{-2} t$,	$v_n(t) = \ln^{-1} t$	if $\beta < \beta_c$, $d/\alpha = 1$;
$u_n(t) = t^{-d/\alpha}$,	$v_n(t) = 1$	if $\beta < \beta_c$, $d/\alpha \in (1, \infty)$,

and λ_0, $C_n(x,y)$, $C_n(x)$ are some positive constants.

The limit theorem (without any restrictions on the variance of the walk jumps)

If $\beta > \beta_c$ then, in the sense of convergence of all moments,

$$\lim_{t\to\infty} \mu_t(y)e^{-\lambda_0 t} = \xi\psi_0(y),$$

$$\lim_{t\to\infty} \mu_t e^{-\lambda_0 t} = \xi,$$

where ξ is a non-degenerate random variable such that $E_x\xi^n = C_n^d(x)$ and

$$\psi_0(x) = \lambda_0 G_{\lambda_0}(x,0).$$

Moreover, under the condition $\beta_n \leq const \cdot n! n^{n-1}$ the moments $C_n(x)$ uniquely determine the distribution of ξ, so that the results are also valid in the sense of convergence in distribution.

Carleman's condition

Let $\{m_n\}$ be the moment sequence of a random variable $X \geq 0$, and

$$\sum_{n=1}^{\infty} m_n^{-1/(2n)} = \infty$$

Then both the distribution function F and the random variable X are uniquely determined by the moments $\{m_n\}$, or, as commonly stated, the random variable X is M-determinate.

One more significant example of the use is a limit theorem for the random branching walks theory: a random variable arises as a limit of other random variables and its moment asymptotic behaviour is known. In this case, some tool is needed to check whether the distribution of this limiting variable is unique or not.

Fréchet–Shohat Theorem

It is worth noting that to know if a distribution is M-determinate or M-indeterminate is of interest by itself. Moreover, the M-determinacy property is essential in the proof of limit theorems. It is appropriate to recall the following theorem.

Theorem (Fréchet–Shohat)

Let F_N, $N = 1, 2, \ldots$, be a sequence of distribution functions such that for each of them all moments are finite and the following limits exist:

$$\lim_{N\to\infty} m_{n,N} = \lim_{N\to\infty} \int x^n \, dF_N(x) = m_n, \qquad n = 1, 2, \ldots .$$

Then the following two statements are true:

(i) *$\{m_n\}$ is a moments sequence of some distribution function, say F_*;*

(ii) *if $\{m_n\}$ uniquely determines F_*, the weak convergence $F_N \xrightarrow{d} F_*$, $N \to \infty$, holds.*

Irregular growth of the moments

In the case $\beta \le \beta_{\mathrm{c}}$, the growth of the moments for μ_t and $\mu_t(y)$ appears to be irregular with respect to the number n of the moment.

This means that the behavior of the random variables μ_t and $\mu_t(y)$ as $t \to \infty$ substantially differs from the behavior of the moments.

Let $\nu_t(0)$ be the number of particles, which visited the origin during the time t.

For $\beta = \beta_c = 0$ (a case of recurrent critical BRWs) we obtain

$$\nu_t(0) = \int_0^t \mathbf{I}_{\{X(s) \in \mathbf{0}\}}\, ds.$$

Recurrent critical BRWs with a finite variance of jumps

Let **Assumption 1** be valid, $\beta_c = 0$, and $x \in [0, \infty)$. If $d = 1$ then

$$\lim_{t\to\infty} \mathbf{P}\left[\frac{2\nu_t(0)}{\sqrt{\pi}\,\int_0^t m_1(s,0,0)\,ds} \le x\right] = \frac{1}{\sqrt{\pi}} \int\limits_0^x e^{-\frac{t^2}{4}}\,dt.$$

If $d = 2$ then

$$\lim_{t\to\infty} \mathbf{P}\left[\frac{\nu_t(0)}{\int_0^t m_1(s,0,0)\,ds} \le x\right] = 1 - e^{-x}.$$

Recurrent critical BRWs. The case of infinite variance of jumps

Let **Assumption 2** be valid, $\beta_c = 0$, and $x \in [0, \infty)$.

If $d = 1$ and $1 < \alpha < 2$ then

$$\lim_{t\to\infty} \mathbf{P}\left[\frac{\nu_t(0)}{\Gamma(2-1/\alpha)\int_0^t m_1(s,0,0)\,ds} \le x\right] = \mathbf{P}\left(\zeta_{1-\frac{1}{\alpha}} < x\right),$$

where r.v. $\zeta_{1-\frac{1}{\alpha}}$ has for $\psi = 1 - \frac{1}{\alpha}$ the density of the Mittag-Leffler distribution:

$$g_\psi(y) = \frac{1}{\pi\psi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!}\sin(\pi\psi j)\,\Gamma(\psi j+1)y^{j-1}.$$

If $d = 1$ and $\alpha = 1$ then

$$\lim_{t\to\infty} \mathbf{P}\left[\frac{\nu_t(0)}{\int_0^t m_1(s,0,0)\,ds} \le x\right] = 1 - e^{-x}.$$

One of the contemporary trends

is to investigate the limit behavior of $\mu(t, x, y)$ when **both coordinates**, t and y, may vary, that is to undertake **the spatio-temporal analysis** of the evolution of the system.

Zones of various types of asymptotics of the first moments in a critical branching environment

For $n = 1$ and $\beta_c = 0$ we have the equation $\partial m_1/\partial t = \mathcal{H}_0 m_1$ with the initial conditions $m_n(0, x, y) = \delta_y(x)$.

Consider a special case of a simple symmetric random walk $H_0 = \varkappa\Delta$, $\varkappa > 0$, where $(\Delta u)(z) := \sum_{|z'-z|=1}(u(z') - u(z))$, $u \in l^p(\mathbf{Z}^d)$.

The moment $m_1(t, z) := m_1(t, x, y) = m_1(t, 0, y - x)$ can be represented as follows:

$$m_1(t,z) = \frac{e^{\sum_{j=1}^{d}\left(-\frac{z_j^2}{4\varkappa t}+\frac{z_j^4}{192\varkappa^3 t^3}-\frac{z_j^6}{2560\varkappa^5 t^5}+\mathcal{O}\left(\frac{z_j^8}{t^7}\right)\right)}}{(4\pi\varkappa t)^{\frac{d}{2}}\sqrt[4]{\prod_{j=1}^{d}\left(1+\frac{z_j^2}{4\varkappa^2 t^2}\right)}}\left(1+\nu_d(2\varkappa t, z)\right) \qquad (1)$$

here $\nu_d(2\varkappa t, z) \to 0$ as $t^2 + z_j^2 \to \infty$. To study the different zones of the asymptotic behavior of the $m_1(t, z)$, we will need to know the scale of variation of $m_1(t, z)$ as $t \to \infty$ and $z \to \infty$, where z takes values of order t^α with various $\alpha \geq 0$.

Zone $c_1 t^{1/2} \leq |z| \leq c_2 t^{1/2}$

From (1), for a fixed z, we have the asymptotic equality

$$m_1(t, z) \sim (4\pi\varkappa t)^{-\frac{d}{2}}, \quad t \to \infty. \tag{2}$$

In view of (1), representation (2) holds not only for a fixed z, but also for all z satisfying $|z|\, t^{-1/2} \to 0$. In particular, representation (2) holds for all z such that $|z| \leq c\, t^{\alpha}$, where $0 \leq \alpha < \frac{1}{2}$. From (1) it follows, in particular, that

$$m_1(t, z) \sim e^{-\frac{|z|^2}{4\varkappa t}} (4\pi\varkappa t)^{-\frac{d}{2}}, \quad t \to \infty \tag{3}$$

if $|z|$ has the same growth order as $t^{1/2}$, i.e., for some $c_1, c_2 > 0$ satisfying condition $c_1 t^{1/2} \leq |z| \leq c_2 t^{1/2}$.

Zone of moderate deviations of the random walk

In view of (1), representation (3) holds not only for z satisfying the condition $c_1 t^{1/2} \leq |z| \leq c_2 t^{1/2}$ for some $c_1, c_2 > 0$, but also for all z satisfying $c_1 t^{1/2} \leq |z|$ and $\sum_{j=1}^{d} z_j^4 \, t^{-3} \to 0$, where the second relation is equivalent to the condition $|z| \, t^{-3/4} \to 0$ as $t \to \infty$. In particular, representation (3) holds for all z such that $c_1 t^{1/2} \leq |z|$ and $|z| \leq c \, t^{\alpha}$, where $\frac{1}{2} \leq \alpha < \frac{3}{4}$. So, relation (3) already includes the case of moderate deviations of the random walk.

In the case where the variable $|z|$ has the same growth order as $t^{3/4}$, from (1) under the condition $c_1 t^{3/4} \leq |z| \leq c_2 t^{3/4}$ with arbitrary $c_1, c_2 > 0$ we have the asymptotic representation

$$m_1(t,z) \sim e^{-\frac{|z|^2}{4\varkappa t} + \sum_{j=1}^{d} \frac{(z_j)^4}{192\varkappa^3 t^3}} (4\pi\varkappa t)^{-\frac{d}{2}}, \quad t \to \infty. \tag{4}$$

Note that there exists a constant B such that, for $|z| \leq B \, t^{3/4}$,

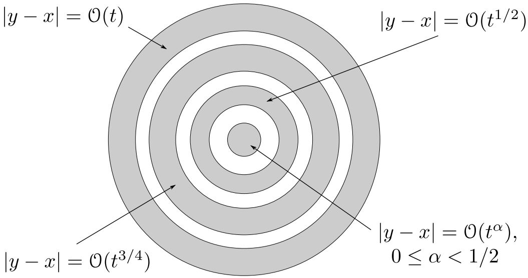
$$m_1(t,z) \geq C_d e^{-B_d \sqrt{t}} t^{-\frac{d}{2}}, \quad C_d > 0, \quad B_d > 0, \quad t \to \infty, \tag{5}$$

where C_d and B_d are some constants depending on the dimension of the lattice.

Zone of moderate deviations of the random walk

Another appeal to (1) shows that representation (4) holds not only for z satisfying the condition $c_1 t^{3/4} \le |z| \le c_2 t^{3/4}$ for some $c_1, c_2 > 0$ and as $t \to \infty$, but also for z satisfying the relations $c_1 t^{3/4} \le |z|$ and $\sum_{j=1}^{d} z_j^6\, t^{-5} \to 0$ as $t \to \infty$, of which the second one is equivalent to the condition $|z|\, t^{-5/6} \to 0$ as $t \to \infty$. In particular, representation (4) holds for all x and y satisfying the relations $c_1 t^{3/4} \le |z|$ and $|z| \le c\, t^{\alpha}$, where $\frac{3}{4} \le \alpha < \frac{5}{6}$. It should be noted that (5) ceases to hold in the case where $c_1 t^{\alpha} \le |z| \le c_2 t^{\alpha}$, but $\frac{3}{4} < \alpha < \frac{5}{6}$.

Zones of various types of the asymptotics of $m_1(t, z)$



Zone $|z| = |y - x|$

Finally, to analyze the setting when $|z|$ has the same growth order as t, we will use not representation (1), but rather a more precise formula, which implies the asymptotic representation

$$m_1(t,z) \sim \frac{e^{-\sum_{j=1}^{d} y_j \operatorname{arcsinh}\left(\frac{z_j}{2\varkappa t}\right) + 2\varkappa t \sum_{j=1}^{d}\left(\sqrt{1+\frac{z_j^2}{4\varkappa^2 t^2}}-1\right)}}{(4\pi\varkappa t)^{d/2} \sqrt[4]{\prod_{j=1}^{d}\left(1+\frac{z_j^2}{4\varkappa^2 t^2}\right)}}, \quad t \to \infty$$

for some $c_1, c_2 > 0$ under the condition $c_1 t \leq |z| \leq c_2 t$. By (1) for $\alpha = 1$ we have the inequality $m_1(t,z) \geq F_d e^{-K_d t} t^{-\frac{d}{2}}$ for some $F_d, K_d > 0$.

Structure of the particle population in a critical branching environment with a given initial number of particles

Consider a lattice population model, that is a random field $n(t,\cdot)$ of particles on $\mathbf{Z}^d$, $d \geq 1$, where $n(t,y)$ is the number of particles at the point $y \in \mathbf{Z}^d$ at the time moment $t \geq 0$. Let $n(0,y) \equiv 1$ for every $y \in \mathbf{Z}^d$. The spatio-temporal evolution of the field includes the migration and birth-death processes. As usual, we exclude interaction between particles.

The total population $n(t,y)$ at a point $y \in \mathbf{Z}^d$ is the sum of independent subpopulations:

$$n(t,y) = \sum_{x \in \mathbf{Z}^d} n(t,x,y),$$

where $n(0,x,y) = \delta(y-x)$, and $n(0,y) \equiv 1$.

For the moment analysis of the field $n(t,y)$ one can use the forward Kolmogorov equations for the correlation functions

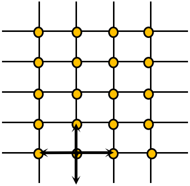
$$K_t(x_1,\ldots,x_m) = En(t,x_1)\cdots n(t,x_m),$$

Homogeneous environment. Generalization for $f(u) := \sum_{n=0}^{\infty} b_n u^n$

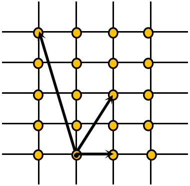
$\mathbf{Z}^d$, d≥1,

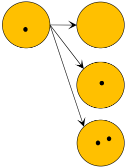
t=0

κΔ_0+V(·)



A+V(·)





``death'' or ``jump''

nothing

``birth''

Model and Results. Shortly

Each particle of the population dies in the interval $(t, t+dt)$ with probability $b_0 dt$, where b_0 is the mortality intensity, or generate $n \neq 1$ offsprings with probability $b_n\, dt + o(dt)$. We suggest that the underlying random walks has a finite variance. We call $\beta = f'(1) = \sum_{n=0}^{\infty} nb_n$ in this model the birth rate. Further we consider only the critical branching process: $\beta = b_0$ at every lattice point.

The total population $n(t, y)$ at a point $y \in \mathbf{Z}^d$ is the sum of independent subpopulations:

$$n(t, y) = \sum_{x \in \mathbf{Z}^d} n_x(t, y), \qquad n_x(t) = \sum_{y \in \mathbf{Z}^d} n_x(t, y),$$

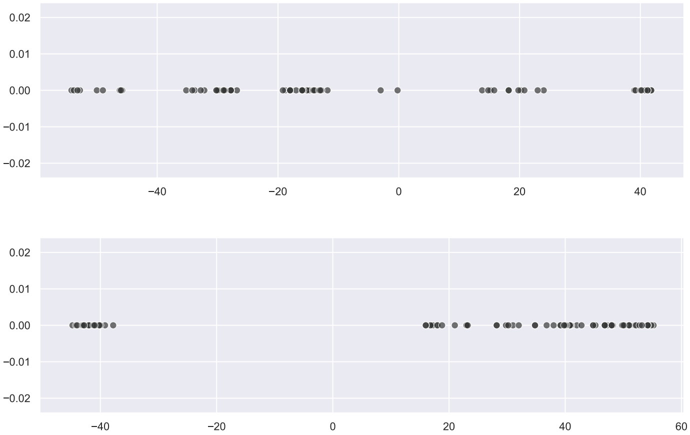
where $n_x(0, y) = \delta(y - x)$, and $n(0, y) \equiv 1$. If $t \to \infty$ and $s > 0$ then

$$P\left\{ \frac{n_x(t)}{\beta t + 1} > s \middle| n_x(t) > 0 \right\} \to e^{-s}.$$

For $t \to \infty$ we have

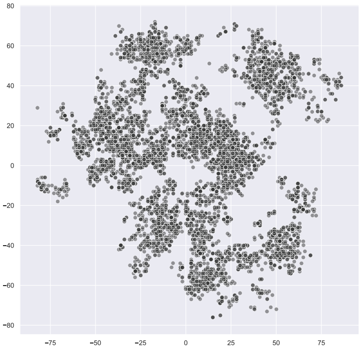
$$\mathbf{E}[n(t, x, y) | n_x(t) > 0] \sim \beta t p(t, x, y).$$

Clusters on $\mathbb{Z}$



The total population demonstrates the high level of intermittency: large clusters (that is clusters of the diameter $O(\sqrt{t})$) are separated by the empty intervals of the length $O(t)$. Each of such clusters contains about t particles.

Clusters on $\mathbf{Z}^2$



Here the typical size of the subpopulation and the typical distance between points $x_{i,t}$ both have order $\sqrt{t}$. However, the population still has fairly large gaps.

Branching Random Walks in Random Environments

The Anderson Hamiltonian. Random Environment

Much attention in the theory of random environment, in particular in the context of localization problem (see, e.g., Carmona and Lacroix (1990)), has been devoted to the study of spectral properties of the operator

$$\varkappa\Delta + \mathcal{V}, \quad \varkappa > 0,$$

where Δ (as above, is the discrete Laplacian on $\mathbf{Z^d}$) and the operator $\mathcal{V}$ are acting as follows

$$(\Delta\psi)(x) = \frac{1}{2d} \sum_{|x'-x|=1} \left(\psi(x') - \psi(x)\right), \quad (\mathcal{V}\psi)(x) := V(x)\psi(x),$$

and where the potential $V(x) = V(x, \omega)$, $x \in \mathbf{Z^d}$, $d \geq 1$, is a random function determined by the random branching medium.

The Parabolic Anderson Problem. Random Environment

Random perturbations play an important role in the intermittency theory for the so-called parabolic Anderson localization problem (Gärtner et al., Gärtner and Molchanov 2009), where the Anderson localization is a general wave phenomenon that applies to the transport of electromagnetic waves, acoustic waves, quantum waves, spin waves, etc.

The Parabolic Anderson Problem. Random Environment

If in BRW

- the transport of particles is governed by the law of **a simple SRW**,
- the random branching environment is defined by **random** birth and death intensities **at every lattice point**,

then **the expected total number of particles** (the first order moment) satisfies the Cauchy problem with a random potential:

$$\partial_t m_1(t,x) = \varkappa(\Delta m_1(t,\cdot))(x) + (\mathcal{V} m_1(t,\cdot))(x), \quad m_1(0,x) \equiv 1.$$

Here $\partial_t := \partial/\partial t$ stands for the partial derivative with respect to the time t.

Intermittency Phenomenon

Mathematical theory of intermittency in random environments was developed by Zeldovich, Molchanov, Gärtner, Carmona et al.

It has been discovered that the evolution of the field $m_1(t,x)$ leads to the formation of highly irregular spatio-temporal structures, characterized by **the generation of rare high peaks on a low-profile background**.

The Study of Intermittency

The study of intermittency in the works of J. Gärtner, S. Molchanov are based on **asymptotic analysis of the moments** $\langle m_1 \rangle$ obtained by averaging the random moment m_1 over medium's realizations, where the angular brackets $\langle \cdot \rangle$ denote expectation with respect to the random environment.

For instance, the second moments grow much faster than the squared first moments, the fourth moments behave in the same way with respect to the squared second moments, and so on:

$$\langle m_1^2 \rangle \gg \langle m_1 \rangle^2, \qquad \langle m_1^4 \rangle \gg \langle m_1^2 \rangle^2, \qquad \dots .$$

Main Objectives. Preliminary Remarks

The evolution of the mean number of particles m_1 in a **nonhomogeneous random environment** is determined by the operator

$$\mathcal{A} + V(0)\mathcal{V}_0,$$

where the RW generator A is a bounded self-adjoint operator in $l^2(\mathbf{Z^d})$ and

$$(\mathcal{V}_0 u)(x) := \delta_0(x)u(x), \qquad x \in \mathbf{Z}^d,$$

while $V(0)$ is a random variable characterizing the source intensity.

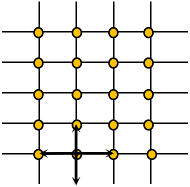
Objectives

1. to extend the results obtained earlier for the discrete Laplacian $\varkappa\Delta$ in the model of BRW in a spatially homogeneous branching random environment to a wider class of symmetric RW with the RW generator A, in particular, to solve the Cauchy problem for the operator $\mathcal{A}+\mathcal{V}$,

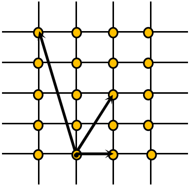
Objective 1. Random Homogeneous Environment. Generalization for SBRW

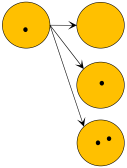
$\mathbf{Z}^d$, $d \geq 1$, **t=0**

$\kappa\Delta_0 + V(\cdot)$



$A + V(\cdot)$





``death'' or ``jump''

nothing

``birth''

Objectives

2 to study the long-time behavior of the moments $\langle m_n^p \rangle$ ($p \geq 1$, $n \geq 1$) for the local and total particle populations for BRW in a nonhomogeneous branching random environment, in particular to solve the Cauchy problem for the operator $\mathcal{A} + V(0)\mathcal{V}_0$,

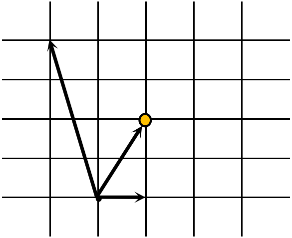
3 to determine **conditions** enabling the long-time behavior of the moments $\ln \langle m_n^p \rangle$ for the numbers of particles at an arbitrary site of the lattice and on the entire lattice to coincide for both models of BRW in spatially **homogeneous** and **nonhomogeneous** random environments,

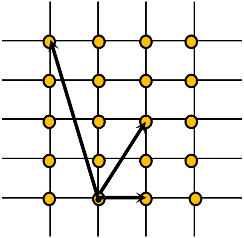
4 to construct examples where the distributions of the random potential V satisfy these **conditions**.

Objectives 2-3. Random Nonhomogeneous and Homogeneous Environments.

$\mathbf{Z}^d$, $d \geq 1$, $t=0$

SBRW: $A+V(0)\Delta_0$ $A+V(\cdot)$





BRW in Homogeneous Random Environments

Suppose now that a branching random environment is formed by pairs of non-negative random variables, $\xi(x) := (\xi^-(x), \xi^+(x))$, $x \in \mathbf{Z}^d$, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

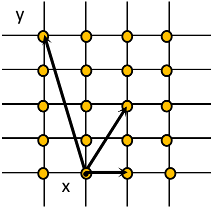
The sample point $\omega \in \Omega$ represents sample realizations of the field $\xi(\cdot)$. In particular, we can assume that $\Omega = (\mathbf{R}_+^2)^{\mathbf{Z}^\mathbf{d}}$.

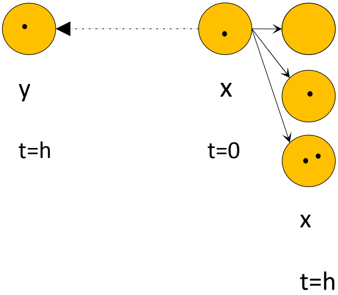
The expectation with respect to the probability measure $\mathbf{P}$ will be denoted by angular brackets, $\langle \cdot \rangle$.

We assume that the random field ξ is spatially homogeneous, that is, the distribution $\mathbf{P}$ of the field is invariant with respect to translations $x \mapsto x + y$, $x, y \in \mathbf{Z}^\mathbf{d}$ (see, e.g., S. Albeverio, L. Bogachev, S. Molchanov and E. Yarovaya (2000)).

BRW in a Random Homogeneous Environment

$\mathbf{Z}^d$, $d \geq 1$, $A+V(\cdot)$, $t=0$





at x: ``death'' $\xi^-(x)h+o(h)$ or ``jump'' $a(x,y)h+o(h)$

nothing $1+a(0)h-(\xi^+(x)+\xi^-(x))h+o(h)$

``birth'' $\xi^+(x)h+o(h)$

Homogeneous Environment. The Moments Equations

Let

$$V(x) := \xi^{+}(x) - \xi^{-}(x), \quad x \in \mathbf{Z^d}.$$

Then the moment functions $m_n(t,x,y)$, $m_n(t,x)$ satisfy the chain of linear differential equations

$$\begin{aligned} \frac{dm_1}{dt} &= \mathcal{A}m_1 + \mathcal{V}m_1, \\ \frac{dm_n}{dt} &= \mathcal{A}m_n + \mathcal{V}m_n + \mathcal{K}^{+}g_n[m_1,\dots,m_{n-1}], \quad n = 1,2,\dots \end{aligned}$$

with the initial conditions $m_n(0,\cdot,y) = \delta_y(\cdot)$, $m_n(0,\cdot) \equiv 1$, where

$$g_n[m_1,\dots,m_{n-1}] := \sum_{i=1}^{n-1} \binom{n}{i} m_i m_{n-i}, \quad n \geq 2,$$

$$(\mathcal{K}^{+}u)(x) := \xi^{+}(x)u(x), \qquad x \in \mathbf{Z}^d.$$

BRW in Nonhomogeneous Random Environments

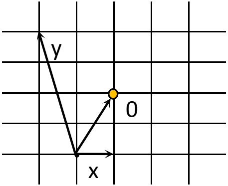
Suppose now that a branching random environment is formed by only the one pair of non-negative random variables, $\xi(0) := (\xi^{-}(0), \xi^{+}(0))$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. It is assumed that $\Omega = \mathsf{R}_{+}^{2}$.

In this case, the random environment is spatially non-homogeneous, since the branching medium formed of birth-and-death process only at the origin of the lattice.

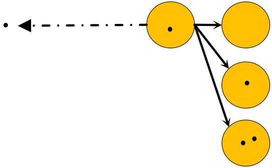
BRW in a Random Nonhomogeneous Environment

$\mathbf{Z}^d$, d≥1, t=0

$A+V(0)\Delta_0$



At the origin:



Nonhomogeneous Environment

Let

$$V(0) := \xi^+(0) - \xi^-(0).$$

Here, the first-order moments satisfy the following homogeneous equation in operator form

$$\frac{dm_1}{dt} = \mathcal{A}m_1 + V(0)\mathcal{V}_0 m_1.$$

Evolutionary Operators of BRW in Random Environments

Simple SBRW in a Homogeneous Random Environment: $\varkappa\Delta + \mathcal{V}$

$$\Downarrow$$

SBRW in a Homogeneous Random Environment:

$$\mathcal{A} + \mathcal{V}$$

$$\Downarrow$$

SBRW in a Nonhomogeneous Random Environment: $\mathcal{A} + V(0)\mathcal{V}_0$

Kolmogorov's backward equation

Suppose that x_t is a continuous-time "jumping" trajectory of a continuous-time symmetric random walk on $\mathbf{Z^d}$ with the generator A, and E_x is the expectation under the condition that the random walk starts from x.

Theorem (Kolmogorov's backward equation)

Define $p(t,x,y) = E_x \delta_y(x_t)$. *Then* $p(t,\cdot,y) \in l^2(\mathbf{Z}^\mathbf{d})$ *for each* $t > 0$ *and*

$$\frac{dp}{dt} = \mathcal{A}p, \quad p(0,x,y) = \delta_y(x), \tag{6}$$

where the right-hand side is interpreted as a linear operator $\mathcal{A}$ *applied to the function* $x \mapsto p(t,x,y)$ *by the formula:*
$(\mathcal{A}p(t,\cdot,y))(x) = \sum_{x'} a(x,x')p(t,x',y)$.

Moreover, if $p^*(t,x,y)$ *satisfies the Cauchy problem* (6), *then* $p^*(t,x,y) = p(t,x,y)$ *with* $p(t,x,y) = E_x \delta_y(x_t)$.

Theorem (J. Gärtner and S. Molchanov, 1990)

Assume that $V(x)$ i.i.d. Then the Cauchy problem has a unique non-negative solution if

$$\left\langle \left(\frac{V(0)}{\ln_+ V(0)} \right)^d \right\rangle < \infty, \tag{7}$$

where $\ln_+ V(0) := \ln \max(V(0), e)$.

Theorem (Homogeneous Random Environment)

Assume that (7) holds and

$$\begin{aligned} m_1(t,x,y) &= E_x\left[\exp\left(\int_0^t V(x_s)\,ds\right)\delta_y(x_t)\right], \\ m_1(t,x) &= E_x\left[\exp\left(\int_0^t V(x_s)\,ds\right)\right]. \end{aligned}$$

Then $m_1(t,x,y)$ and $m_1(t,x)$ $\mathbf{P}$-a.s. satisfy the Cauchy problem:

$$\frac{dm_1}{dt} = \mathcal{A}m_1 + \mathcal{V}m_1,$$

with the initial conditions $m_n(0,\cdot,y) = \delta_y(\cdot)$ and $m_1(0,\cdot) \equiv 1$, respectively.

Theorem (Nonhomogeneous Random Environment)

Assume that (7) holds for $V(0)$ and

$$\begin{aligned} m_1(t,x,y) &= E_x\left[\exp\left(V(0)\int_0^t \delta_0(x_s)\,ds\right)\delta_y(x_t)\right], \\ m_1(t,x) &= m_1(t,x) = E_x\left[\exp\left(V(0)\int_0^t \delta_0(x_s)\,ds\right)\right]. \end{aligned}$$

Then $m_1(t,x,y)$ and $m_1(t,x)$ $\mathbf{P}$-a.s. satisfy the Cauchy problem:

$$\frac{dm_1}{dt} = \mathcal{A}m_1 + V(0)\mathcal{V}_0 m_1,$$

with the initial conditions $m_n(0,\cdot,y) = \delta_y(\cdot)$ and $m_1(0,\cdot) \equiv 1$, respectively.

Remark

Now we are able to give our main result on the long-time behavior of the moments $\langle m_n^p \rangle$ where $n \in \mathbf{N}$, $p \geq 1$. Under the assumption that the analyzed Cauchy problems $\mathbf{P}$-a.s. have a unique non-negative solutions, the following theorem holds.

Theorem (Homogeneous and Nonhomogeneous Random Environments (2010))

Let $V := V(0)$. Assume that

$$\lim_{t\to\infty} \frac{t}{\ln\left\langle e^{Vt}\right\rangle} = 0.$$

Then for all integer moments $\langle m_n^p \rangle$, where m_n is the solution of the Cauchy problems for BRW in ***homogeneous or nonhomogeneous*** *random environments) with the initial conditions $m_n(0,\cdot,y) = \delta_y(\cdot)$ and $m_n(0,\cdot) \equiv 1$, respectively, we obtain*

$$\lim_{t\to\infty} \frac{\ln\left\langle m_n^p\right\rangle}{\ln\left\langle e^{pnVt}\right\rangle} = 1.$$

Conclusion

In this way, condition

$$\lim_{t\to\infty} \frac{t}{\ln \left\langle e^{Vt} \right\rangle} = 0 \tag{8}$$

appears ensuring that the long-time behavior of the moments $\langle m_n^p \rangle$, $n \geq 1$, for the numbers of particles at arbitrary site of the lattice and on the entire lattice coincide for both models of BRW in spatially **homogeneous and nonhomogeneous random** environments.

Nonhomogeneous non-random environments

If the spectrum of the operator $\mathcal{A} + \beta\Delta_0$ contains a maximum eigenvalue $\lambda > 0$, then both the local numbers of particles and their total number grow exponentially as $t \to \infty$:

$$\lim_{t\to\infty} \mu_t(y)\, e^{-\lambda t} = \xi\psi(y), \quad \lim_{t\to\infty} \mu_t\, e^{-\lambda t} = \xi. \tag{9}$$

Here, $\psi(y)$ is a function and ξ is a non-degenerate random variable. This case is referred to as supercritical. Relations (9) hold in the sense of convergence in distribution. In particular, for the first moment if $\beta > G_0^{-1}(0,0)$, then for $n \in \mathbf{N}$, as $t \to \infty$,

$$m_1(t,x,y) \sim C_1(x,y)e^{\lambda t}, \quad m_1(t,x) \sim C_1(x)e^{\lambda t}.$$

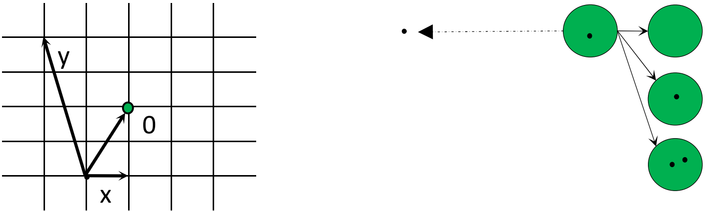
Nonhomogeneous non-random environments

Hence for supercritical BRW in an **nonhomogeneous non-random** environment the analog of (8) has the form

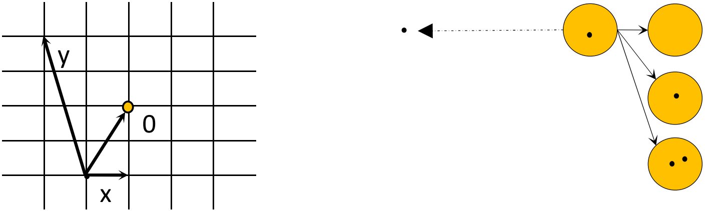
$$\lim_{t\to\infty}\frac{t}{\ln m_1}=\lim_{t\to\infty}\frac{t}{\ln e^{\lambda t}}=\frac{1}{\lambda}$$

and (8) is not valid. The validity of the condition (8) means that the distribution of the potential V has **the tail heavier than exponential**.

Nonhomogeneous non-random environment: $A+\beta\Delta_0$



Nonhomogeneous random environment: $A+V(0)\Delta_0$



“Heavy Tails” of Distributions of the Potential V

Here we construct examples of distributions of the random potential V satisfying the condition

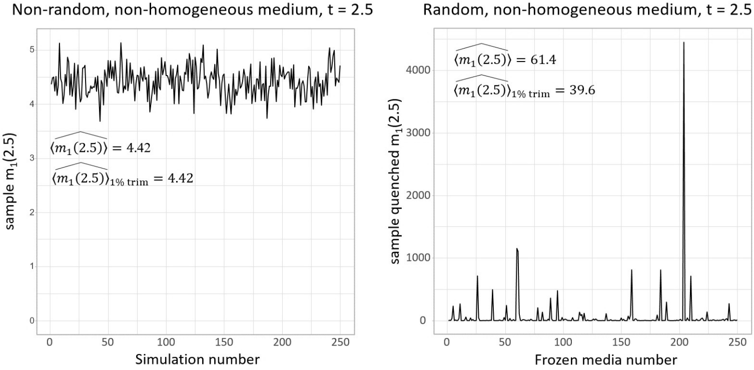
$$\lim_{t \to \infty} \frac{t}{\ln \langle e^{Vt} \rangle} = 0.$$

Remark

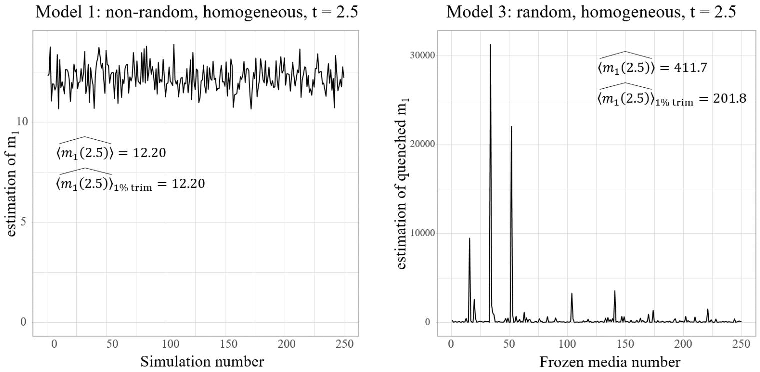
Distributions with “heavy tails” have numerous applications in the catastrophe theory. It can partially be explained by the fact that catastrophes are rare events and their tails decay more slowly than any exponential tail. Therefore these distributions are often used to model disasters and other rare events.

▸ Examples

Simulation on Z. One Source of Branching



Simulation on Z. Sources of branching at every lattice point



1.png

Example: Weibull-type upper tail

We begin with a theorem where the tail of the distribution of the branching potential V has a Weibull type upper tail:

$$\ln \mathbf{P}\{V > r\} \sim -cr^{\gamma}, \quad \gamma > 1, c > 0. \quad r \to \infty, \tag{10}$$

Theorem

Under assumption (10), *we have for every* $p \geq 1$

$$\ln \left\langle e^{pV_t} \right\rangle \sim (\gamma - 1) \left(\frac{pt}{\gamma c^{1/\gamma}} \right)^{\gamma/(\gamma-1)}, \quad t \to \infty.$$

Condition (8) *also holds in this case.*

If $\gamma = 2$, we have an immediate corollary for the case where the upper tail is of **Gaussian type**.

Example: Gumbel-type upper tail

The upper tail of the distribution of the branching potential V has the following form:

$$\ln \mathbf{P}\{V > r\} \sim -\exp(r/c), \quad c > 0. \quad r \to \infty, \tag{11}$$

Theorem

Under assumption (11), *we have for every* $p \geq 1$

$$\ln \left\langle e^{pV_t} \right\rangle \sim cpt \ln t, \quad t \to \infty.$$

Condition (8) *also holds in this case.*

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