On the notion of asymptotic independence of random variables

S. Novikov

SPbSU

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Main problem

In this work we consider the following question:
Under which conditions can two sequences of random variables (X_n) and (Y_n) be considered as asymptotically independent?
Intuitive answer: the joint distributions $P_{(X_n,Y_n)}$ must be close to the products of the corresponding marginal distributions $P_{X_n} \times P_{Y_n}$.

Various definitions of weak merging of measures

Definition (The Levy-Prokhorov metric)

$\pi(P, Q) = \inf\{\epsilon : P(A_\epsilon) \leq Q(A)$ for all closed $A\}$.

There exist at least 3 definitions of weak merging of probability measures P_n, Q_n:
D1 $\pi(P_n, Q_n) \to 0$
D2 $\int h dP_n - \int h dQ_n \to 0$ for each bounded and continuous function h.
D3 $T(P_n) - T(Q_n) \to 0$ for each bounded continuous (in weak-star topology) functional T on the space of probability measures.

D3 $\Rightarrow$ **D2** $\Rightarrow$ **D1**
We will use **D1**, as **D2** is too strong: e.g., $P_n = \delta_n$, $Q_n = \delta_{n+1/n}$ do not satisfy D_2.

Various conditions of asymptotic independence

Let $X_n \in (E_1, \mathcal{E}_1)$, $Y_n \in (E_2, \mathcal{E}_2)$, where E_1, E_2 are complete separable metric spaces, $\mathcal{E}_1$ and $\mathcal{E}_2$ are their Borel σ-algebras.

(**AI-0**) $\mathbb{E}f(X_n)g(Y_n) - \mathbb{E}f(X_n)\mathbb{E}g(Y_n) \to 0, n \to +\infty$ for each two **uniformly** continuous, bounded $f : E_1 \to \mathbb{R}^1$, $g : E_2 \to \mathbb{R}^1$

(**AI-1**) $\int h(x,y)dP_{(X_n,Y_n)}(x,y) - \int h(x,y)dP_{X_n} \times P_{Y_n}(x,y) \to 0$ for each bounded **uniformly** continuous function $h : E_1 \times E_2 \to \mathbb{R}^1$

(**AI-2**) $\forall\ A \in \mathcal{E}_1$, $B \in \mathcal{E}_2$
$|P_{(X_n,Y_n)}(A \times B) - P_{X_n}(A)P_{Y_n}(B)| \to 0, n \to +\infty$

(**AI-3**) $\sup\limits_{A \in \mathcal{E}_1, B \in \mathcal{E}_2} |P_{(X_n,Y_n)}(A \times B) - P_{X_n}(A)P_{Y_n}(B)| \to 0,\ n \to +\infty$

(**AI-4**) $||P_{(X_n,Y_n)} - P_{X_n} \times P_{Y_n}||_{var} \to 0,\ n \to +\infty$
It is easy to see that **AI-4** $\Rightarrow$ **AI-3** $\Rightarrow$ **AI-2** $\Rightarrow$ **AI-0**.

Also remark that **AI-1** is equivalent to $\pi(P_{(X_n,Y_n)}, P_{X_n} \times P_{Y_n}) \to 0$, where π is the Levy-Prokhorov metric.

Weak dependence. Mixing.

Let $X = (\xi_n)_{n\in\mathbb{Z}}$ be a stationary sequence.
Let Y_n be a shifted sequence: $(Y_n)_k = \xi_{n+k}$.
Consider X and Y_n as elements of the space $(E, \mathcal{E})$, where $E = \mathbb{R}^{\mathbb{Z}}$ и $\mathcal{E}$ is the σ-algebra of cylidric sets. T is the Bernoulli shift, μ is the distribution of X(T-invariant).

Mixing

For each $A, B \in \mathcal{E}$ $\mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$, that is:

$$P\{X \in A, Y_n \in B\} \to P\{X \in A\}P\{X \in B\}$$

By taking $X_n = X$ and using stationarity, we obtain:

$$P\{X_n \in A, Y_n \in B\} - P\{X_n \in A\}P\{Y_n \in B\} \to 0.$$

That is, **mixing** is a particular case of **AI-2**.

Strong mixing

In the setting of the previous example let $X_n : \Omega \to \mathbb{R}^{-\mathbb{N}}$ be the restriction of $X = (\xi_k)_{k\in\mathbb{Z}}$ to $\{..., -1, 0\}$ and let $Y_n : \Omega \to \mathbb{R}^{\mathbb{N}}$ be the restriction X to $\{n, n+1, ...\}$. Let $\mathcal{M}_a^b = \sigma\{\xi_a, ..., \xi_b\}$.

Strong mixing

As $n \to +\infty$:

$$\sup_{A\in\mathcal{M}_{-\infty}^{0}, B\in\mathcal{M}_n^{+\infty}} |P\{A\cap B\} - P\{A\}P\{B\}| \to 0$$

This is equivalent to

$$\sup_{C\in\mathbb{R}^{-\mathbb{N}}, D\in\mathbb{R}^{\mathbb{N}}} |P\{X_n \in C, Y_n \in D\} - P\{X_n \in C\}P\{Y_n \in D\}| \to 0.$$

That is, **strong mixing** is a particular case of **AI-3**.

AI under transformations

Let (X_n), (Y_n) be two AI sequences and f, g two functions. Do the transformed sequences $(f(X_n)), (g(Y_n))$ remain AI?

Proposition 6

a) **AI-0**,**AI-1** remains fulfilled if f, g are uniformly continuous.
b) **AI-2** remains fulfilled if f, g are measurable.
c) **AI-3** and **AI-4** remain fulfilled if f_n, g_n are measurable.

Counterexamples

It is not hard to construct counterexamples, which show that **AI-1** does not always imply **AI-2**, and **AI-2** does not imply **AI-3**.

Natural question: are **AI-0** and **AI-1** always equivalent?

It turns out that if (X_n), (Y_n) are not tight, then the answer is negative, and, moreover, the following result is true:

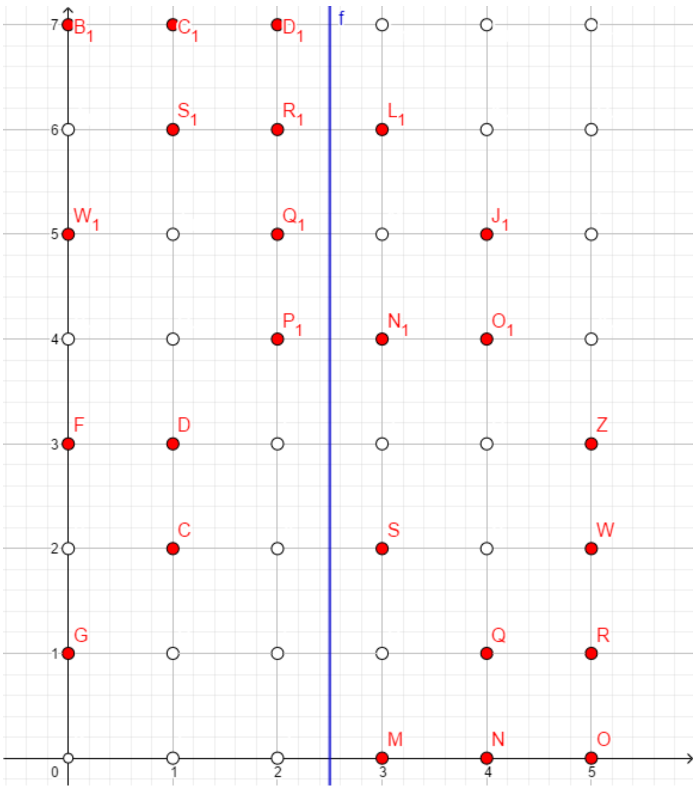
Prop. 1 [S.N., 2019]

Let $E_1 = \mathbb{R}$, $E_2 = \mathbb{R}$, then there exist two sequences (X_n), (Y_n), which satisfy **AI-0**, but not **AI-1**.

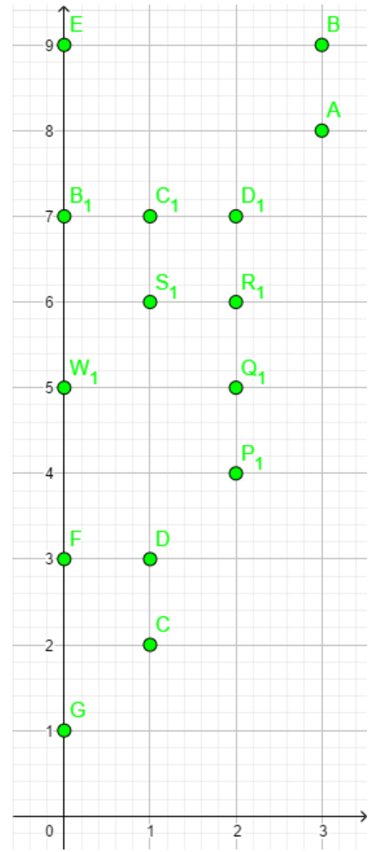
Main counterexample

$$\begin{array}{cccccc}
1/24 & 1/24 & 1/24 & 0 & 0 & 0 \\
0 & 1/24 & 1/24 & 1/24 & 0 & 0 \\
1/24 & 0 & 1/24 & 0 & 1/24 & 0 \\
0 & 0 & 1/24 & 1/24 & 1/24 & 0 \\
1/24 & 1/24 & 0 & 0 & 0 & 1/24 \\
0 & 1/24 & 0 & 1/24 & 0 & 1/24 \\
1/24 & 0 & 0 & 0 & 1/24 & 1/24 \\
0 & 0 & 0 & 1/24 & 1/24 & 1/24
\end{array}$$

This is the distribution of (X_n, Y_n) for $n = 3$. ($2n$ colums, $2^n - 1$ rows: binary codes of 0,...,$2^n - 1$)



$P_{(X_n,Y_n)}, n = 3$



the support of h

Uniform convergence in **AI-0** and **AI-1**

For a metric space M denote by $BL_1(M)$ the class of 1-**Lipschitz** functions on M, which do not exceed 1 in absolute value.

Fact [Y. Davydov, V. Rotar', 2009]

If $(P_n), (Q_n)$ are two sequences of probability distributions on M, and for each **uniformly** continuous bounded $h : M \to \mathbb{R}$ we have $\int h(x)d(P_n - Q_n) \to 0$, then

$$\sup_{h \in BL_1(M)} \left|\int h(x)d(P_n - Q_n)\right| \to 0,\ n \to +\infty$$

This, obviously, implies uniform convergence in **AI-1** with respect to $h \in BL_1(E_1 \times E_2)$. A similar fact is true for **AI-0**:

Theorem 1 [S.N., 2020]

If $(X_n), (Y_n)$ satisfy **AI-0**, then

$$\sup_{f \in BL_1(E_1),\ g \in BL_1(E_2)} |\mathbb{E}f(X_n)g(Y_n) - \mathbb{E}f(X_n)\mathbb{E}g(Y_n)| \to 0, n \to +\infty$$

The case, when one or both (P_{X_n}), (P_{Y_n}) are tight

1) [S.N., 2020] If **at least one** of the sequences (P_{X_n}), (P_{Y_n}) is tight, then **AI-0** and **AI-1** are equivalent.

With the help of relative compactness one can obtain:
2) [S.N., Y.Davydov, 2019] If **both** the sequences (P_{X_n}), (P_{Y_n}) are tight, and $X_n \in \mathbb{R}^k$, $Y_n \in \mathbb{R}^m$, then **AI-0**, **AI-1** are equivalent to the following condition on characteristic functions:

$$\phi_{(X_n,Y_n)}(t,s) - \phi_{X_n}(t)\phi_{Y_n}(s) \to 0,\ n \to +\infty$$

Remark that [S.N., 2020] part 2) does not hold when only **one** of (P_{X_n}), (P_{Y_n}) is tight!

Case $X_n,\ Y_n \in \mathbb{R}^\infty$

Consider $\mathbb{R}^\infty$ with the metric $d((x_1, x_2, ...), (y_1, y_2, ...)) = \sum\limits_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1+|x_k - y_k|}$. In this case **AI-0** and **AI-1** can be checked in terms of finite-dimensional distributions (here $\pi_k : \mathbb{R}^\infty \to \mathbb{R}^k$ is the projection on the first k coordinates):

Theorem 2 [S.N., 2020]

The sequences (X_n), (Y_n) satisfy **AI-1** if and only if for each $k > 0$ $(\pi_k X_n)$, $(\pi_k Y_n)$ satisfy **AI-1**. Same for **AI-0**.

Case $X_n,\ Y_n \in \mathbb{R}^\infty$

It is easy to see that analogous statements for **AI-2**,**AI-3**,**AI-4** are false:

Prop. 2 [S.N., 2020]

There exist two sequences of random elements (X_n), (Y_n) of $\mathbb{R}^\infty$ such that $(\pi_k X_n)$, $(\pi_k Y_n)$ satisfy **AI-4** for each k, but (X_n), (Y_n) do not satisfy **AI-2**.

On the other hand, if we demand that **AI-2**, **AI-3** or **AI-4** holds **"uniformly"** with respect to k, then **AI-2**,**AI-3** or **AI-4** respectively will hold for (X_n), (Y_n).

Gaussian case

Let $X_n = (X_n^{(1)}, ..., X_n^{(k)}) \in \mathbb{R}^k$, $Y_n = (Y_n^{(1)}, ..., Y_n^{(m)}) \in \mathbb{R}^m$.

1) Remark that **AI-1** does not imply **AI-2** even in the Gaussian case: let $X \sim \mathcal{N}(0,1)$, then $X_n = \frac{X}{n}$, $Y_n = \frac{-X}{n}$ do not satisfy **AI-2** (take $A = B = [0, \infty)$), but satisfy **AI-1**.

2) If (X_n), (Y_n) **are tight**, then **AI-1** is equivalent to $cov\{X_n^{(i)}, Y_n^{(j)}\} \to 0$ for $1 \le i \le k$, $1 \le j \le m$.

Gaussian case

With the help of explicit formulas on the total variation distance between Gaussian measures one can obtain a criterion of merging of Gaussian measures in terms of characteristic functions:

Lemma 1 [S.N., 2020]

Let $P_n = \mathcal{N}(0, K_n)$, $Q_n = \mathcal{N}(0, L_n)$ be Gaussian distributions on $\mathbb{R}^d$. If $\sup\limits_{|x|\leq 1} |\phi_{P_n}(x) - \phi_{Q_n}(x)| \to 0, \quad n \to +\infty$, and there exists $\epsilon > 0$ such that the matrices $L_n - \epsilon I_d$ are positive semidefinite for all n, then

$$||P_n - Q_n||_{var} \to 0, \quad n \to +\infty.$$

Theorem 3 [S.N., 2020]

If $X_n \in \mathbb{R}^k$, $Y_n \in \mathbb{R}^m$, (X_n, Y_n) is **Gaussian** for all n; (X_n), (Y_n) satisfy **AI-0** and there exists $\epsilon > 0$ such that $cov(X_n) - \epsilon I_k$ and $cov(Y_n) - \epsilon I_m$ are positively semidefinite for all n, then (X_n), (Y_n) satisfy **AI-4**.

Gaussian case

Using approximation with nondegenerate distributions, we obtain:

Lemma 2 [S.N., 2020]

Let $P_n = \mathcal{N}(0, K_n)$, $Q_n = \mathcal{N}(0, L_n)$ be Gaussian distributions on $\mathbb{R}^d$. If

$$\sup_{|x|\leq 1} |\phi_{P_n}(x) - \phi_{Q_n}(x)| \to 0, \quad n \to +\infty,$$

then $\pi(P_n, Q_n) \to 0$, where π is the Levy-Prokhorov metric.

Theorem 4 [S.N., 2020]

Suppose that $X_n \in \mathbb{R}^k$, $Y_n \in \mathbb{R}^m$, (X_n, Y_n) is Gaussian for all n and (X_n), (Y_n) satisfy **AI-0**. Then (X_n), (Y_n) satisfy **AI-1**.

Gaussian case

With the help of characteristic functions one can also deduce:

Theorem 5 [S.N., 2020]

Suppose that $X_n \in \mathbb{R}^k$, $Y_n \in \mathbb{R}^m$, (X_n, Y_n) is Gaussian for all n and (X_n), (Y_n) satisfy **AI-3**. Then (X_n), (Y_n) satisfy **AI-4**.

Theorem 6 [S.N., 2020]

Suppose that $X_n \in \mathbb{R}^k$, $Y_n \in \mathbb{R}^m$, (X_n, Y_n) is Gaussian for all n and (X_n), (Y_n) satisfy **AI-2**. Then (X_n), (Y_n) satisfy **AI-3**.

Hence, in the Gaussian case we have **AI-0** $\Leftrightarrow$ **AI-1**, **AI-2** $\Leftrightarrow$ **AI-3** $\Leftrightarrow$ **AI-4**.

Gaussian case in $\mathbb{R}^\infty$

Finally, on combining the results of the previous two sections, we obtain:

Corollary 1 [S.N., 2020]

Suppose that X_n, $Y_n \in \mathbb{R}^\infty$, and (X_n, Y_n) is Gaussian for all n, in addition $(X_n), (Y_n)$ satisfy **AI-0**, then they satisfy **AI-1**.

Conclusion

We can propose the following directions of further research:
1. Consider other conditions of the type

$$\int_{E_1} f dP_{X_n} \int_{E_2} g dP_{Y_n} - \int_{E_1 \times E_2} (f \times g) dP_{(X_n, Y_n)} \to 0$$

for all functions f, g from some classes $\mathcal{F}_1$, $\mathcal{F}_2$.

2. Find sufficient conditions for **AI** of the following type:
"If $(f(X_n))$, $(g(Y_n))$ are asymptotically independent for all f, g from some classes $\mathcal{F}_1$, $\mathcal{F}_2$, then (X_n), (Y_n) are also asymptotically independent"

3. Instead of $\mathbb{R}^\infty$ one can consider other spaces (for example, $C[0,1]$).

4. Get results about equivalence of **AI** in some more general case than Gaussian.

Conclusion

Finally, there are two additional open questions:

5. Does **AI-0** imply **AI-1**, when $(X_n,\ Y_n)$ is Gaussian, and $X_n,\ Y_n$ are elements of some separable Hilbert space H?

6. Is merging of probability measures with respect to the metric

$$d'_{BL}(P,Q) = \sup_{f\in BL_1(E_1),\, g\in BL_1(E_2)} \left| \int f(x)g(y)dP - \int f(x)g(y)dQ \right|$$

equivalent to merging of probability measures with respect to the metric

$$\pi'(\mu,\nu) = \inf\{\varepsilon : \mu(A\times B) \le \nu(A^\varepsilon \times B^\varepsilon) + \varepsilon,\ \nu(A\times B) \le \mu(A^\varepsilon \times B^\varepsilon) + \varepsilon \quad \textit{for each closed } A,B\}?$$

Thank you for your attention!