On conditional characteristics of PSI-processes and their sums

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Definition of PSI-processes

Define PSI-process as follows

$$\psi(t) = \xi_{\Pi(\Lambda(t))} = \xi_{\Pi(\lambda t)}, \text{ where } t \geq 0 \tag{1}$$

1. $\vec{\xi} = (\xi_0, \xi_1, \dots)$ are i.i.d. random variables.
2. $\Pi(t)$ – standard Poisson process.
3. $\Lambda(t)$ – accumulated intensity. We consider $\Lambda(t) = \lambda t$, $\lambda > 0$.

Fact: $\psi(t)$ is stationary process.

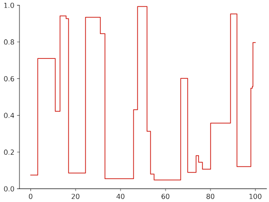


Figure 1: Typical trajectories of $\psi(t)$, $\psi(0) \in U_{[0,1]}$

Sum of PSI-processes (Gaussian case)

Let $\psi_1, \psi_2, \ldots$ be independent identically distributed PSI-processes, $\mathbb{E}\psi_j(0) = 0$, $\mathbb{D}\psi_j(0) = 1$, $\forall j \in \mathbb{N}, s \geq 0$,

$$Z_N(s) = \tfrac{1}{\sqrt{N}} \sum_{j=1}^{N} \psi_j(s) \underset{N\to\infty}{\overset{f.d.d}{\longrightarrow}} Z(s). \tag{2}$$

It is known that $Z(s)$ is an Ornstein–Uhlenbeck process.

Lévy's field on $\mathbb{R}^2$

Definition (Lévy field)

Let $(\mathbb{R}^2, \mathcal{B}, \mu)$ be an Euclidean space with a Lebegue measure, $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space, and let Z_t be a Levy process.
We consider $\mathbb{A}$ as a set of all measurable subsets of $\mathbb{R}^2$ with a finite measure μ, and Ω_0 as a set of all random variables in considering probability space. Finally we define a Lévy Field $L : \mathbb{A} \to \Omega$ by the following conditions:

1. $\forall A \in \mathbb{A}, L(A) \overset{d}{=} Z_{\mu(A)}$ and $L(A) = 0$ a.s. in case $\mu(A) = 0$;
2. If $\{A_i\}_{i=1}^{n}$ is a set of pairwise disjoint elements of $\mathbb{A}$
   1. $L(A_i)$ are jointly independent;
   2. $\bigcup\limits_{i=1}^{\infty} A_i \in \mathbb{A} \;\Rightarrow\; \sum_{i=1}^{\infty} L(A_i) = L\left(\bigcup\limits_{i=1}^{\infty} A_i\right)$.

Upstairs representation of the Ornstein–Uhlenbeck process

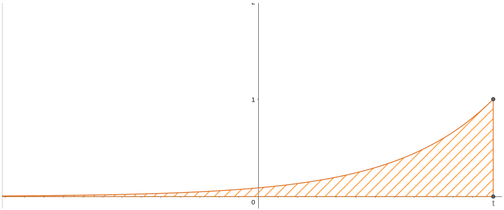
Definition (Upstairs representation of the Ornstein–Uhlenbeck process over the field L)

Let L be a Lévy field, and

$$K_t = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 < t, 0 < x_2 < e^{-(t-x_1)}\}, \; t \in \mathbb{R}. \tag{3}$$

Then $\overline{U}(t) = L(K_t), \; t \geq 0$ is a Upstairs representation of Ornstein–Uhlenbeck process over L field.

We are going to use **U. O-U** abbreviation.



Sum of PSI-processes (Series scheme)

Let L be a Lévy field and $X_0 = L(A),\ \mu(A) = 1$. Then X_0 is a random variable with infinitely divisible distribution. Therefore, there exists a series scheme $(X_i^n)_{i=1}^{k_n}$.

1. X_i^n are i.i.d. ($\forall i\ X^n \stackrel{d}{=} X_i^n$).
2. $\sum_{i=1}^{k_n} X_i^n \underset{n\to\infty}{\Rightarrow} X_0$ (weak convergence).

For simplicity we suppose that $k_n = n$.

Then $\psi^n(s)$ — a PSI-process with $\xi_i^n \stackrel{d}{=} X^n$ as a subordinated sequence. Let $\psi_i^n(s)$ be independent copies of $\psi^n(s)$.

$$U_N(s) = \sum_{j=1}^{N} \psi_j^N(s) \underset{N\to\infty}{\overset{f.d.d}{\to}} \overline{U}(s), s \geq 0 \tag{4}$$

Upstairs O-U as a limit of sums of PSI-processes

Lévy–Khintchine representation of characteristics function of Z_t:

$$\mathbb{E}(exp(iuZ_t)) = exp(t\Phi(u)),\ \Phi : \mathbb{R}^d \to \mathbb{C} \tag{5}$$

1. Brownian motion: $\Phi(u) = \frac{1}{2}u^2,\ u \in \mathbb{R}$;
2. Gamma process: $\Phi(u) = -ln(1 - iu),\ u \in \mathbb{R}$;
3. Poisson process: $\Phi(u) = e^{iu} - 1,\ u \in \mathbb{R}$.

Upstairs O-U as a limit of sums of PSI-processes

Theorem (Podluzniy, Rusakov, 2020)

Denote ϕ^n — a characteristics function of ξ^n. Let ϕ^n satisfy the following condition

$$\phi^N(u) = 1 + \frac{\Phi(u)}{N} + \bar{\bar{o}}\left(\frac{1}{N}\right),\ \forall u \in \mathbb{R} \text{ where } N \to \infty, \tag{6}$$

where Φ is such that characteristics function of ξ_0 is equal $\phi(u) = e^{\Phi(u)}$. Then

$$\sum_{i=1}^{N} \psi_i^N(s) \underset{N\to\infty}{\overset{f.d.d}{\to}} \overline{U}(s),\ s \geq 0. \tag{7}$$

Where $\overline{U}(s)$ is an Upstairs O-U under field L.

Conditional characteristics

Let's add parameter $t \in [0, 1]$ to considering sums.

$$U_N(s,t) = \sum_{j=1}^{[Nt]} \psi_j^N(s) \underset{N\to\infty}{\overset{f.d.d}{\to}} \overline{U}(s,t) \tag{8}$$

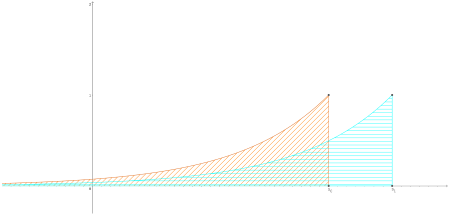


Figure 3: When $t = 1$ is fixed.

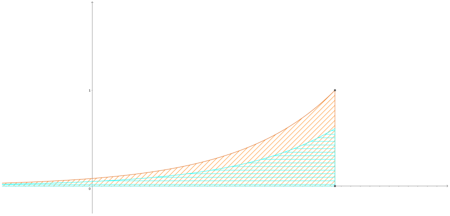


Figure 4: When s is fixed.

Conditional characteristics (Gaussian case)

We will perform permutations for $\psi_1, \psi_2, \dots$ in descending order by a moment of first jump in the corresponding Poisson processes.
In case $s = s_0$ is fixed we still will have a Brownian motion by t as a limit in gaussian case.
Let's denote this Brownian motion as $W^{s_0}(t)$
The following relations holds:

- $W^{s_0}(t) = W^0(t),\ t \in [0, e^{-\lambda s_0}]$
- $W^{s_0}(t) - W^{s_0}(e^{-\lambda s_0})$ is independent from $W^0(t)$ for $t \in [e^{-\lambda s_0}, 1]$

a is a landing point of Brownian bridge.

Conditional characteristics (Gaussian case)

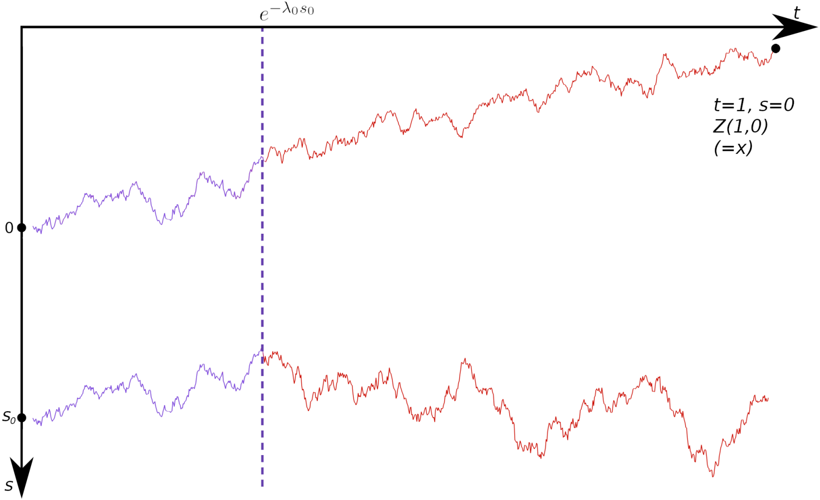


Figure 5

Conditional characteristics (Gaussian case)

Let's denote a Brownian bridge that has value a at time τ by

$$W^{\{a,\tau\}}(s),\ s \in [0,\tau]$$

Let also $W(s),\ s \in [0,\infty)$ be a standart Brownian motion.
Let $W_\tau^* = \sigma\left\{W(s) - \frac{s}{\tau}W(\tau), s \in [0,\tau]\right\}$ be a Brownian bridge filtration.
Then the following relations hold:

1. W_τ^* is independent from $\sigma\{W(s), s \in [\tau,\infty)\}$.
2. $W^{\{a,\tau\}}(s) \overset{d}{=} W(s) - \frac{s}{\tau}W(\tau) + \frac{s}{\tau}a$.

Conditional characteristics (Gaussian case)

$$\begin{aligned}
\mathbb{E}\{\overline{U}(s,1)|\overline{U}(0,1)=x\} = \mathbb{E}\{W^s(1)|W^0(1)=x\} = \\
= \mathbb{E}\{W^s(e^{-\lambda s}) + (W^s(1) - W^s(e^{-\lambda_0 s}))|W^0(1)=x\} = \\
= \mathbb{E}\{W^0(e^{-\lambda s})|W^0(1)=x\} + \mathbb{E}\{W^s(1) - W^s(e^{-\lambda_0 s})|W^0(1)=x\} = \\
= xe^{-\lambda s}
\end{aligned} \tag{9}$$

$$\begin{aligned}
\mathbb{D}\{\overline{U}(s,1)|\overline{U}(0,1)=x\} = \mathbb{D}\{W^s(1)|W^0(1)=x\} = \\
= \mathbb{D}\{W^s(e^{-\lambda s}) + (W^s(1) - W^s(e^{-\lambda_0 s}))|W^0(1)=x\} = \\
= \mathbb{D}\{W^0(e^{-\lambda s})|W^0(1)=x\} + \mathbb{D}\{W^s(1) - W^s(e^{-\lambda_0 s})|W^0(1)=x\} = \\
= e^{-\lambda s} - e^{-2\lambda s} + 1 - e^{-\lambda s} = 1 - e^{-2\lambda s}
\end{aligned} \tag{10}$$

Conditional characteristics (Gamma case)

Let's denote a Gamma bridge that has value a at time τ by

$$\Gamma^{\{a,\tau\}}(s),\ \ s \in [0,\tau],\ \ a > 0$$

Let also $\Gamma(s),\ \ s \in [0,\infty)$ be a standart Gamma process.
Let $\Gamma^*_\tau = \sigma\left\{\dfrac{\Gamma(s)}{\Gamma(\tau)}, s \in [0,\tau]\right\}$ be a Gamma bridge filtration.
Then the following relations hold:

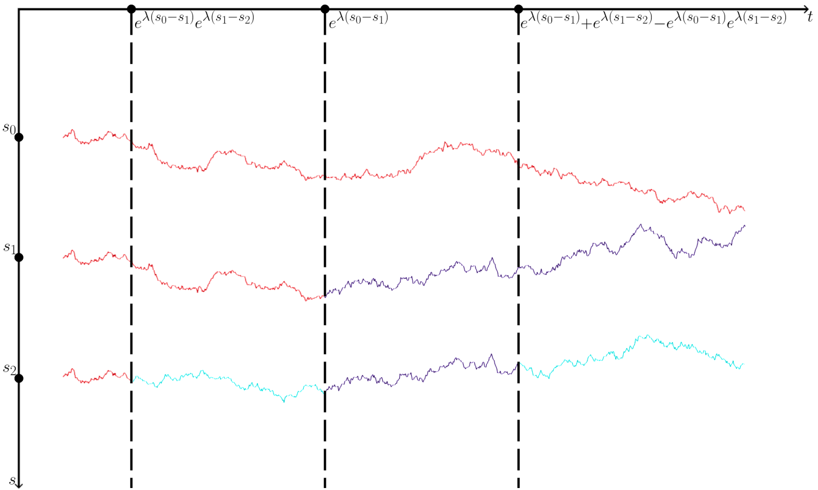
1. Γ^*_τ is independent from $\sigma\left\{\Gamma(s), s \in [\tau,\infty)\right\}$ (Yor, Emery, 2004).
2. $\Gamma^{\{a,\tau\}}(s) \overset{d}{=} \dfrac{\Gamma(s)}{\Gamma(\tau)}a$ (Vershik, 1999).

So gamma bridge would have a Beta distribution. This allows us to calculate conditional mathematical expectation and conditional variance for U. O-U under Gamma field.

Conditional characteristics (Gamma case)

$$\mathbb{E}(\overline{U}(s,1)|\overline{U}(0,1)=x)=\mathbb{E}(\Gamma^s(1)|\Gamma_0(1)=x)=1+e^{-\lambda_0 s}(x-1) \quad (11)$$

$$\mathbb{D}(\overline{U}(s,1)|\overline{U}(0,1)=x)=\mathbb{D}(\Gamma^s(1)|\Gamma_0(1)=x)= \\ =(1-e^{-\lambda_0 s})\left(\frac{x^2 e^{-\lambda_0 s}}{2}+1\right) \quad (12)$$



$e^{\lambda(s_0-s_1)}e^{\lambda(s_1-s_2)}$
$e^{\lambda(s_0-s_1)}$
$e^{\lambda(s_0-s_1)}+e^{\lambda(s_1-s_2)}-e^{\lambda(s_0-s_1)}e^{\lambda(s_1-s_2)}$
t
s_0
s_1
s_2
s

Figure 6

Martingale-Markov property

Lemma

Let $0 \le s_0 < s_1 < s_2$.

$$\mathbb{E}(\overline{U}(s_2) \mid \overline{U}(s_1) = z_1, \overline{U}(s_0) = z_2) = \mathbb{E}(\overline{U}(s_2) \mid \overline{U}(s_1) = z_1) \tag{13}$$

Proof.

Denote $\theta_k = e^{\lambda(s_k - s_{k+1})}$, $k \in \mathbb{N} \cup \{0\}$. Let's reorder pieces of trajectory $\overline{U}(s_2, t)$ to set first ones that were not replaced during transition from s_1 to s_2. Part that was replaced would be an Levy process that is independent from replaces ones. On the first two parts we will have the following mathematical expectation due to Harness property:

$$\theta_0\theta_1 z_0 + \theta_1(z_1 - z_0\theta_0) = \theta_1 z_1 \tag{14}$$

So as it was obtained it doesn't depend on z_0. □

Martingale-Markov property

Theorem

Let $0 < s_0 < s_1 < \ldots < s_m$.

$$\mathbb{E}(\overline{U}(s_m) \mid \overline{U}(s_{m-1}) = z_{m-1}, \ldots, \overline{U}(s_0) = z_0) \\ = \mathbb{E}(\overline{U}(s_m) \mid \overline{U}(s_{m-1}) = z_{m-1}) \quad (15)$$

Proof.

Proof is based on the telescopic property for conditional mathematical expectation. □

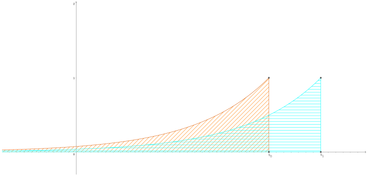
U. O-U mathematical expectation

Lemma

$$\mathbb{E}(\overline{U}(s) \mid \overline{U}(0) = z) = ze^{s} + \mathbb{E}Z(1 - e^{-s}), \tag{16}$$

where $Z(s)$ is the corresponding Lévy process.
Note that Z is a an arbitrary Lévy process with mathematical expectation.

Proof.



The painted twice area has a mathematical expectation ze^{s} due to Harness property (double-side martingale). Painted once in the cian color area has a mathematical expectation equal to $\mathbb{E}Z(1 - e^{-s})$. □

U. O-U variance

Hypotheses (proof is in progress):

1. There is a Martingale-Markov property for variance of U. O-U as well;
2. In case there is a variance of Lévy bridge it looks like $C(t - t^2)$;
3. In case there is a variance of U. O-U it looks like

$$\mathbb{D}(\overline{U}_s \mid \overline{U}_0 = z) = C(e^{-s} - e^{-2s}) + \mathbb{D}Z(1 - e^{-s}) \tag{17}$$

(17) is proved for the Gamma $\overline{U}$, where $C = \frac{z^2}{2}$.

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